The rank condition for discrete series reps

joint with Krötz, Opdam and Schlichtkrull.

Let G be real reductive alg. group

H alg. subgroup.

$Z = G/H$

Assume $Z = G/H$ admits a pos. G-inv
Radon measure.

How does $L^2(Z)$ decompose into irred.?

$G = SO(2)$ Fourier series
G cpt Peter-Weyl
$\Big\}$ discrete

$G = \mathbb{R}^n, \; H = \{0\}$

$$L^2(\mathbb{R}^n) \simeq \int_{i\mathbb{R}^n}^{\oplus} \mathbb{C}_\xi \, d\xi$$

In general mixture of discrete and cont.:

Def: Z is called real spherical if a minimal parabolic subgroup admits open orbit Z

$\iff \exists$ min. parabolic subalg $\mathfrak{p}$ s.t. $\mathfrak{p} + \mathfrak{h} = \mathfrak{g}$

<u>Ex</u>: 1) $G \times G / \mathrm{diag}(G) \simeq G$

2) Z symmetric

$\exists\, \sigma \in \mathrm{Aut}(G)$, $\sigma^2 = \mathrm{id}$

$H \underset{\text{open}}{\subseteq} G^\sigma$ for ex. $H = K$ max cpt.

3) $$\begin{tikzcd} GL(n,\mathbb{R}) \arrow[r, phantom, "\underset{\text{symm}}{\subseteq}"] \arrow[rr, bend right, "\text{real spherical}"'] & SO(n,n) \arrow[r, phantom, "\underset{\text{symm}}{\subseteq}"] & SO(n,n+1) \end{tikzcd}$$

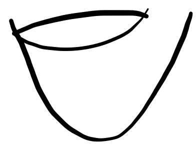
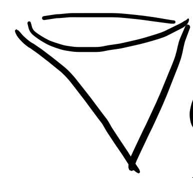
pairs $(\mathfrak{g},\mathfrak{h})$ with $\mathfrak{h}$ real spherical and reduct.

are classified

4) If $\mathfrak{h}$ real sph

$$\overline{\mathrm{Ad}(G)\mathfrak{h}} \subseteq Gr(\mathfrak{g}, \dim \mathfrak{h})$$

consists of real spherical subalgs.

5) $SL(2,\mathbb{R})$  $G/SO(2)$  G/N

Delorme, Knop, Krötz, Schlichtkrull

Description of Pl decomp in terms of Bernstein

Descr. upto multiplicities, discrete spectrum

Def: An irred unitary rep π belongs to the discrete series of reps for Z, iff $\mathrm{Hom}_G(\pi, L^2(Z)) \neq 0$, i.e., iff π can be realized on closed subspace of $L^2(Z)$

<u>Question</u>: When do d.s. reps exist?

Let K be a max. cpt. subgroup.

1) Harish-Chandra

If $Z = G \times G/\mathrm{diag}(G)$, then

$\exists$ d.s. reps $\iff \mathrm{rk}(G) = \mathrm{rk}(K)$

2) Flensted-Jensen, Oshima-Matsuki

If $\mathcal{Z}$ symm. ($H \overset{\text{open}}{\subseteq} G^\sigma$), $\sigma(K) = K$

then $\exists$ d.s. rep $\Leftrightarrow$ rk(G/H) = rk$(K/K \cap H)$

3) $\mathcal{D}, K, K, S$,

If Z is real spherical, then

$\exists$ d.s. reps for Z exist iff

$\underbrace{\operatorname{int}}_{\operatorname{in} \mathfrak{h}^{\perp}} \{ X \in \underbrace{\mathfrak{h}^{\perp}}_{\text{w.r.t. } G \text{ inv. non deg bilinear form}} \mid X \text{ elliptic} \} \neq \emptyset$

Conjecture : if and only if

Infinitesimal characters.

Let π be irred unitary rep

$\mathcal{U}(\mathfrak{g})$ acts on V_π^∞ dense subspace of V_π

The action of $Z(\mathfrak{g})$ the center of $U(\mathfrak{g})$ commutes with action $\overset{\text{Schur}}{\Longrightarrow}$ $Z(\mathfrak{g})$ acts by scalars.

$\exists$ character $\chi_\pi : Z(\mathfrak{g}) \to \mathbb{C}$ s.t.

$z \cdot v = \chi_\pi(z) \cdot v$ $\quad z \in Z(\mathfrak{g})$; $v \in V_\pi^\infty$

If $\mathfrak{t}_{\mathbb{C}} \subseteq \mathfrak{g}_{\mathbb{C}}$ is Cartan subalg,
then via H-C iso

char's of $Z(\mathfrak{g}) \longleftrightarrow \mathfrak{t}_{\mathbb{C}}^{*} / W_{\mathbb{C}}$

Weyl group.

Let $\mathfrak{a} \subseteq \mathfrak{g}$ be max ab subalg consisting of X s.t. ad(X) is diag with real e.v.

Let $\mathfrak{t} \subseteq \mathfrak{z}_{\mathfrak{g}}(\mathfrak{a})$ be max. ab subalg consisting X s.t. ad(X) is diag$/_{\mathbb{C}}$ and with purely imaginary e.v.

Then $(\mathfrak{a} \oplus \mathfrak{t})_{\mathbb{C}} \subseteq \mathfrak{g}_{\mathbb{C}}$ is Cartan subalg.

To every irred unitary rep, attach

$$[\lambda_\pi] \in (\mathfrak{a} \oplus \mathfrak{t})^*_{\mathbb{C}} / W_{\mathbb{C}}$$

<u>Thm</u> (KKOS.1)

Assume Z is real spherical

$\exists\, W_{\mathbb{C}}$-invariant lattice

$\Lambda \subseteq (\mathfrak{a}+i\mathfrak{t})^*$, so that

π d. s. rep for $Z \Rightarrow \lambda_\pi \in \Lambda$

Let $\sigma : (\mathfrak{a} + \mathfrak{t})^*_{\mathbb{C}} \longrightarrow (\mathfrak{a} + \mathfrak{t})^*_{\mathbb{C}}$

be conjugation w.r.t. $(\mathfrak{a} + \mathfrak{t})^*$

If π is irred. unitary, then

$$[-\lambda_\pi] = [\sigma \lambda_\pi]$$

$\pi^\vee$ is realized on $\overline{V}_\pi$

If π is discrete series rep for $\mathfrak{k}$ then

$\exists\, w \in W_{\mathbb{C}}$ so that $\boxed{-\sigma(\lambda_\pi) = w\lambda_\pi}$

Now group case:

$$rk(G) = rk(K) \iff \boxed{-\sigma\big|_{(\mathfrak{a}+i\mathfrak{t})^*} \in W_{\mathbb{C}}}$$

<u>Def</u>: $\lambda_t(\sigma + i\tau)^*$ is <u>strongly regular</u> if its normalizer in $\langle W_{\mathbb{C}}, -\sigma \rangle$ is trivial

If π is discrete series rep with strongly reg. inf. char, then $rk(G) = rk(K)$

Idea: Given a d.s. rep π, construct π' d.s. rep with $\lambda_{\pi'}$ is strongly reg.

Proof KKOS2.

Zuckerman translation principle

Let π be a d.s., F finite dim rep.

$\pi \otimes F$ has finite length and inf char's are known

In general $\pi \otimes F \not\hookrightarrow L^2(G)$

However if λ_π is very large and

F is small, then $\pi \otimes F \hookrightarrow L^2(G)$

First construct π' with $\lambda_{\pi'} = N\lambda_\pi$

for large $N \in \mathbb{N}$

G/H , H compact

$\mathrm{rk}(G) = \mathrm{rk}(K)$

What is the condition on H. st.

G/H admits d.s. rep. ?