Asymptotic behaviour for the Heat Equation on Noncompact Symmetric Spaces

Effie Papageorgiou

University of Crete,
Department of Mathematics and Applied Mathematics

June 15, 2021

Outline

Asymptotic convergence of heat eq. solutions

Problem

Consider an integrable solution $u(x,t)$ of the heat equation. What does it eventually look like, i.e.,

$$u(\cdot,t) \xrightarrow{L^1} ?$$

Asymptotic convergence of heat eq. solutions

Problem

Consider an integrable solution $u(x,t)$ of the heat equation. What does it eventually look like, i.e.,

$$u(\cdot,t) \xrightarrow{L^1} ?$$

Does geometry affect the answer to the question?

Work in progress with J.-Ph. Anker, H.-W. Zhang.

The heat kernel

X complete Riemannian manifold, Ricci curvature bdd from below.

The heat equation

$$\begin{cases} \partial_t u(x,t) &= \Delta_x u(x,t), \quad x \in X, \ t > 0 \\ u(x,0) &= f(x), \quad x \in X. \end{cases} \tag{1}$$

Heat kernel $h_t(x,y)$: smallest positive fundamental sol. to heat eq.
Solution to (1):

$$u(x,t) = \int_X h_t(x,y) f(y) \mathrm{dvol}(y).$$

Note: $\int_X h_t(x,y)\mathrm{dvol}(y) = 1.$

Heat asymptotics: euclidean case

$$\mathbb{R}^n : \quad h_t(x) = (4\pi t)^{-\frac{n}{2}} e^{-\frac{|x|^2}{4t}}.$$

Heat asymptotics: euclidean case

$$\mathbb{R}^n: \quad h_t(x) = (4\pi t)^{-\frac{n}{2}} e^{-\frac{|x|^2}{4t}}.$$

$r(t)$: positive increasing function s.t. $\frac{r(t)}{\sqrt{t}} \to +\infty$ as $t \to +\infty$.
Then,

$$\int_{x\in\mathbb{R}^n:\ |x|\geq r(t)} h_t(x)dx = c_n \int_{\frac{r(t)}{\sqrt{t}}}^{+\infty} r^{n-1} e^{-\frac{r^2}{4}} dr = O((\frac{r(t)}{\sqrt{t}})^{-\infty}) \to 0, \text{ as } t \to +\infty.$$

In other words: heat kernel on $\mathbb{R}^n$ asymp. concentrates in ball $B(0, r(t))$.

The euclidean case

Proposition

Let $f \in L^1(\mathbb{R}^n)$ *and* $\int_{\mathbb{R}^n} f(x)dx = M \neq 0$. *Then,*

$$\|u(t,x) - Mh_t(x)\|_{L^1} \to 0, \quad \text{as } t \to \infty.$$

The euclidean case

Proof.

Euclidean heat kernel: $h_t(x) = (4\pi t)^{-\frac{n}{2}} e^{-\frac{|x|^2}{4t}}$. Then,

$$
\begin{aligned}
u(t,x) - Mh_t(x) &= \int_{\mathbb{R}^n} h_t(x-y)f(y)dy - \int_{\mathbb{R}^n} f(y)dy h_t(x) \\
&= \int_{\mathbb{R}^n} (h_t(x-y) - h_t(x))f(y)dy.
\end{aligned}
$$

MVT &

$$
\int_{\mathbb{R}^n} |\nabla h_t(x)| dx \lesssim \frac{1}{\sqrt{t}} \int_0^\infty r e^{-r^2} dr \to 0.
$$

□

Heat asymptotics: hyperbolic space

Davies - Mandouvalos:

$$\mathbb{H}^n: \; h_t(r) \asymp t^{-n/2}(1+r)(1+t+r)^{\frac{n-3}{2}} e^{-\frac{(n-1)^2}{4}t - \frac{n-1}{2}r - \frac{r^2}{4t}}, \; r = d(x,o).$$

Heat asymptotics: hyperbolic space

Davies - Mandouvalos:

$$\mathbb{H}^n: \; h_t(r) \asymp t^{-n/2}(1+r)(1+t+r)^{\frac{n-3}{2}} e^{-\frac{(n-1)^2}{4}t - \frac{n-1}{2}r - \frac{r^2}{4t}}, \; r = d(x,o).$$

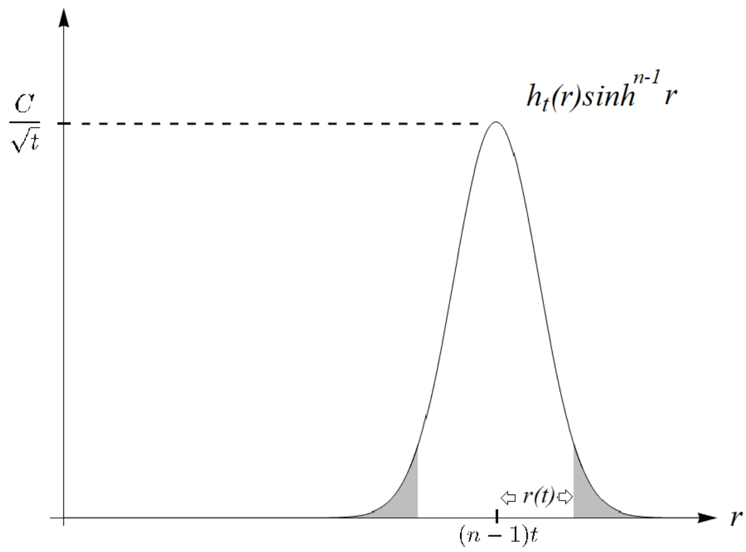
Heat kernel $\times$ vol. element is a Gaussian, w. exp.term $e^{-\frac{(r-(n-1)t)^2}{4t}}$.
Then (Davies),

$$\int_{|r-(n-1)t|>r(t)} h_t(r) \sinh^{n-1} r dr \to 0, \quad \frac{r(t)}{\sqrt{t}} \to +\infty.$$

Heat asymptotics: hyperbolic space

Heat on hyperbolic space concentrates in the annulus

$$(n-1)t - r(t) \leq r \leq (n-1)t + r(t), \quad \frac{r(t)}{\sqrt{t}} \to +\infty.$$



The hyperbolic space case: euclidean approach?

Problem

The heat kernel gradient does not always provide time decay.

The hyperbolic space case: Euclidean approach?

Problem

The heat kernel gradient does not always provide time decay.

Example

In $\mathbb{H}^3$, $h_t(r) = (4\pi t)^{-\frac{3}{2}}\, e^{-t}\, \frac{r}{\sinh r}\, e^{-\frac{r^2}{4t}}$.
So,

$$\frac{d}{dr} h_t(r) = h_t(r)\left(\frac{1}{r} - \coth r - \frac{r}{2t}\right)$$

with no decay in time in the critical region

$$2t - r(t) \leq r \leq 2t + r(t).$$

Definitions

- $X = G/K$ symmetric space of noncompact type:
  -G noncompact semisimple Lie group (connected, finite center)
  -K maximal compact subgroup
- $\mathfrak{g} = \text{lie}(G)$, $\text{ad}(X)(Y) = [X, Y]$
- Killing form $B(X, Y) = tr(\text{ad}X \text{ad}Y)$
- $\mathfrak{g}$ semisimple $\iff$ B non degenerate

Definitions

- $X = G/K$ symmetric space of noncompact type:
  -G noncompact semisimple Lie group (connected, finite center)
  -K maximal compact subgroup
- $\mathfrak{g} = \text{lie}(G)$, $\text{ad}(X)(Y) = [X, Y]$
- Killing form $B(X, Y) = tr(\text{ad}X \text{ad}Y)$
- $\mathfrak{g}$ semisimple $\iff$ B non degenerate
- $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}$, where $\mathfrak{k} = \text{lie}(K)$, $\mathfrak{p} \cong T_e X$.
- $\mathfrak{a} \subset \mathfrak{p}$ maximal abelian subspace, $\mathfrak{a}^*$ dual
- $\langle \cdot, \cdot \rangle$ inner product on $\mathfrak{a}$, norm $|\cdot|$. Same notation on $\mathfrak{a}^*$.
- $\dim \mathfrak{a} = \ell = \text{rank}\, X$.

Root systems

- Root: $\alpha \in \mathfrak{a}^*$ so that $\forall H \in \mathfrak{a}$, $\alpha(H)$ eigenvalue of Lie bracket. From now on: identify $\mathfrak{a}^*$ w. $\mathfrak{a}$.
- Fix $t \in \mathfrak{a}$, choose all α s.t. $\langle \alpha, t \rangle > 0 \rightsquigarrow \alpha$: *positive* root.
- *Simple* root: positive, not sum of two other positive roots.
- Positive Weyl chamber: $\mathfrak{a}^+ = \{\lambda : \langle \alpha, \lambda \rangle > 0, \ \forall \alpha \text{ simple}\}$.

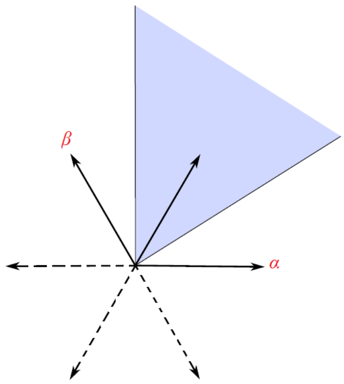


Figure: Root system A_2

Bottom of the spectrum

- ρ=half sum of positive roots, counted with multiplicities
- $|\rho|^2$: geometric invariant of X, bottom of the L^2 spectrum.
- In rank one symmetric spaces, $\rho \in \mathbb{R}_+$.

Example

$\mathbb{H}^n(\mathbb{R})$, $\rho = |\rho| = \frac{n-1}{2}$.

Cartan decomposition

Cartan decomposition

$$G = K \underbrace{\exp \overline{\mathfrak{a}^+}}_{\text{"radial" component}} K,$$

$$x = k_1 \exp(H) k_2, \quad H \text{ unique.}$$

On $G/K = K \exp \overline{\mathfrak{a}^+}$: analogue of polar decomposition in $\mathbb{R}^n$.

Cartan decomposition

Cartan decomposition

$$G = K \underbrace{\exp \overline{\mathfrak{a}^+}}_{\text{“radial” component}} K,$$

$$x = k_1 \exp(H) k_2, \quad H \text{ unique.}$$

On $G/K = K \exp \overline{\mathfrak{a}^+}$: analogue of polar decomposition in $\mathbb{R}^n$.

On G: $|x| = |H|$. On G/K, $|x| = |xK|$ distance to the origin.

Haar measure

- Functions on $X = G/K$: functions on G, right-K invariant
- K-bi-invariant functions on G: functions on X, K-invariant on the left, i.e.

$$f(kx) = f(x), \quad \forall k \in K.$$

In rank one, this means just radial: $f = f(r) = f(d(x, o))$.

Haar measure

- Functions on $X = G/K$: functions on G, right-K invariant
- K-bi-invariant functions on G: functions on X, K-invariant on the left, i.e.

$$f(kx) = f(x), \quad \forall k \in K.$$

In rank one, this means just radial: $f = f(r) = f(d(x,o))$.
By Cartan decomposition,

$$\int_G f(g)\, dg = \int_X f(x)\, dx = c \int_{\mathfrak{a}^+} f(\exp H)\delta(H) dH,$$

where $\delta(H) \lesssim e^{2\langle \rho, H\rangle}$, $H \in \overline{\mathfrak{a}^+}$: Jacobian density.

Spherical Fourier transform

Spherical functions: Eigenfunctions of Δ.

$$\Delta\varphi_\lambda = -(|\lambda|^2 + |\rho|^2)\varphi_\lambda, \quad \varphi_\lambda(e) = 1.$$

They play the role of $e_\lambda(x) = e^{i\langle\lambda,x\rangle}$ in Fourier analysis in $\mathbb{R}^n$.

Spherical Fourier transform

- Spherical Fourier transform

$$(\mathcal{H}f)(\lambda) = \int_G f(x)\varphi_{-\lambda}(x)\, dx, \quad \lambda \in \mathfrak{a},$$

for f bi-K-invariant in Schwartz space of G.

- Inversion formula:

$$(\mathcal{H}^{-1}f)(x) = c \int_{\mathfrak{a}} f(\lambda)\varphi_\lambda(x) \frac{d\lambda}{|\mathbf{c}(\lambda)|^2}, \quad x \in G,\ f \in S(\mathfrak{a})^W,$$

where $\mathbf{c}(\lambda)$: Harish-Chandra function.

Heat kernel

$$h_t(x) = \mathcal{H}^{-1}(e^{-t(|\cdot|^2+|\rho|^2)})(x)$$

- Davies-Mandouvalos: real hyperbolic space,

$$h_t(r) \asymp t^{-n/2}(1+r)(1+t+r)^{\frac{n-3}{2}} e^{-\frac{(n-1)^2}{4}t-\frac{n-1}{2}r-\frac{r^2}{4t}}, \quad r = d(x,y).$$

- Anker-Ji, Anker-Ostellari: all symmetric spaces,

$$h_t(\exp H) \asymp t^{-n/2} \left(\prod_{\alpha\in\Sigma_0^+} (1+\langle\alpha, H\rangle)(1+t+\langle\alpha, H\rangle)^{\frac{m_\alpha+m_{2\alpha}}{2}-1} \right)$$

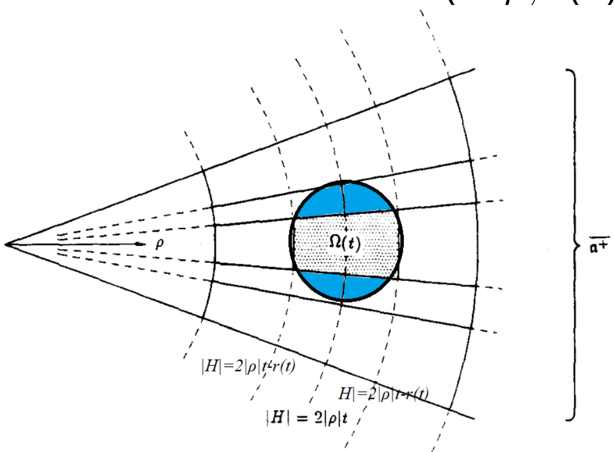
$$\times\, e^{-|\rho|^2 t-\langle\rho,H\rangle-\frac{|H|^2}{4t}}, \quad H \in \overline{\mathfrak{a}^+}.$$

Heat concentration: general case (Anker-Setti)

Using heat kernel estimates:

$$\int_{G \setminus K(\exp B(2t\rho, r(t)))K} dx\, h_t(x) \lesssim \left(\frac{\sqrt{t}}{r(t)}\right)^N \quad \forall N \geq 0,$$

i.e., heat concentrates in bi-K-orbit of ball $B(2t\rho, r(t))$, $\frac{r(t)}{\sqrt{t}} \to +\infty$.



THE RESULT FOR SYMMETRIC SPACES

AIM:

Show that if $f \in L^1(X)$, bi-K-invariant, with $M = \int_X f(x)dx$, then

$$\|u(t,x) - Mh_t(x)\|_{L^1} \to 0, \quad \text{as } t \to \infty.$$

THE RESULT FOR SYMMETRIC SPACES

Remark

Sufficient to show the result for C_c^∞ bi-K-invariant initial data.

Approxim. argument, since $C_c^\infty(K\backslash G/K)$ dense in $L^1(K\backslash G/K)$.

Sketch of the proof

Consider $f \in C_c^\infty(K \backslash G / K)$, supported in $K(\exp B(0,a))K$.

Idea:

- Outside the critical region.
  Estimate
  $$u(t,x) = (f * h_t)(x) = (h_t * f)(x) = \int_G h_t(xy^{-1})\, f(y)\, dy$$
  by reduction to heat kernel asymptotics. Then,
  $$\|u(t,\cdot)\|_{L^1\{\text{non-critical region}\}} \to 0.$$
- Inside the critical region.
  Spherical analysis to estimate difference $u(t,x) - Mh_t(x)$.

Outside the critical region

Solution: $u(t,x) = (h_t * f)(x) = \int_{K(\exp B(0,a))K} dy\ f(y) h_t(xy^{-1}),$

L^1 norm outside critical region:

$$\int_{G \smallsetminus K(\exp B(2t\rho, r(t)))K} dx\, |u(t,x)|$$

$$\leq \int_{K(\exp B(0,a))K} dy\, |f(y)| \underbrace{\int_{G \smallsetminus K(\exp B(2t\rho, r(t)))K} dx\, h_t(xy^{-1})}_{\leq \int_{G \smallsetminus K(\exp B(2t\rho, r(t)-a))K} dz\, h_t(z)} .$$

Outside the critical region

Solution: $u(t,x) = (h_t * f)(x) = \int_{K(\exp B(0,a))K} dy\ f(y) h_t(xy^{-1}),$

L^1 norm outside critical region:

$$\int_{G \smallsetminus K(\exp B(2t\rho, r(t)))K} dx\, |u(t,x)|$$

$$\leq \int_{K(\exp B(0,a))K} dy\, |f(y)| \underbrace{\int_{G \smallsetminus K(\exp B(2t\rho, r(t)))K} dx\, h_t(x y^{-1})}_{\leq \int_{G \smallsetminus K(\exp B(2t\rho, r(t)-a))K} dz\, h_t(z)}.$$

Lemma

$d(xK, yK) \geq |x^+ - y^+|$, $\forall x, y \in G$, where x^+: middle comp. of x in $K \exp \overline{\mathfrak{a}^+} K$.

Outside the critical region

But t large $\Longrightarrow$ $r(t) - a \geq \frac{r(t)}{2}$ $\Longrightarrow$ right hand is also $O\left(\frac{\sqrt{t}}{r(t)}\right)^N$.
In conclusion:

$$\int_{G \smallsetminus K(\exp B(2t\rho, r(t)))K} dx\,|u(t,x)| \lesssim \left(\frac{\sqrt{t}}{r(t)}\right)^N \quad \forall\, N \geq 0,$$

so

$$\int_{G \smallsetminus K(\exp B(2t\rho, r(t)))K} dx\,|u(t,x) - Mh_t(x)| \to 0, \ t \to \infty.$$

Critical region: Spherical analysis

Recall

- Spherical functions:

$$\varphi_\lambda(x) = \int_K e^{(i\lambda-\rho)(H(xk))} = \int_K e^{-(i\lambda+\rho)(H(x^{-1}k))} \rightsquigarrow \varphi_{\pm i\rho}(x) = 1.$$

- Spherical transform:

$$\mathcal{H}f(\lambda) = \int_G f(x)\varphi_{-\lambda}(x)dx \rightsquigarrow \mathcal{H}f(\pm i\rho) = \int_G f(x)dx = \text{mass } M.$$

- If both $f_1 = \mathcal{H}^{-1}m_1,\ f_2 = \mathcal{H}^{-1}m_2$ are bi-K-invariant, then

$$f_1 * f_2 = \mathcal{H}^{-1}m_1 * \mathcal{H}^{-1}m_2 = \mathcal{H}^{-1}(m_1 m_2).$$

- $$h_t(x) = c\int_{\mathfrak{a}} d\lambda |\mathbf{c}(\lambda)|^{-2} e^{-t(|\rho|^2+|\lambda|^2)}\varphi_\lambda(x).$$

Critical region: Spherical analysis

- Write sol. to heat eq. with initial data $f \in C_c^\infty(K \backslash G/K)$ as:

$$
\begin{aligned}
u(t,x) &= (f * h_t)(x) \\
&= \left\{ \mathcal{H}^{-1}(\mathcal{H}f(\cdot)) * \mathcal{H}^{-1}(e^{-t(|\rho|^2+|\cdot|^2)}) \right\}(x) \\
&= \mathcal{H}^{-1}\{\mathcal{H}f(\cdot)e^{-t(|\rho|^2+|\cdot|^2)}\}(x)
\end{aligned}
$$

Critical region: Spherical analysis

- Write sol. to heat eq. with initial data $f \in C_c^\infty(K \backslash G/K)$ as:

$$\begin{aligned} u(t,x) &= (f * h_t)(x) \\ &= \left\{\mathcal{H}^{-1}(\mathcal{H}f(\cdot)) * \mathcal{H}^{-1}(e^{-t(|\rho|^2+|\cdot|^2)})\right\}(x) \\ &= \mathcal{H}^{-1}\{\mathcal{H}f(\cdot)e^{-t(|\rho|^2+|\cdot|^2)}\}(x) \end{aligned}$$

- Consider the difference

$$u(t,x) - Mh_t(x) = const.e^{-t|\rho|^2}\int_{\mathfrak{a}} d\lambda |\mathbf{c}(\lambda)|^{-2} e^{-t|\lambda|^2}\varphi_\lambda(x) \times \underbrace{\{\mathcal{H}f(\lambda) - \mathcal{H}f(i\rho)\}}_{\Omega(\lambda)}. \qquad (2)$$

Critical region: Spherical analysis

- Write sol. to heat eq. with initial data $f \in C_c^\infty(K \backslash G/K)$ as:

$$\begin{aligned} u(t,x) &= (f * h_t)(x) \\ &= \left\{\mathcal{H}^{-1}(\mathcal{H}f(\cdot)) * \mathcal{H}^{-1}(e^{-t(|\rho|^2+|\cdot|^2)})\right\}(x) \\ &= \mathcal{H}^{-1}\{\mathcal{H}f(\cdot)e^{-t(|\rho|^2+|\cdot|^2)}\}(x) \end{aligned}$$

- Consider the difference

$$u(t,x) - Mh_t(x) = const.e^{-t|\rho|^2} \int_{\mathfrak{a}} d\lambda |\mathbf{c}(\lambda)|^{-2} e^{-t|\lambda|^2} \varphi_\lambda(x) \times \underbrace{\{\mathcal{H}f(\lambda) - \mathcal{H}f(i\rho)\}}_{\Omega(\lambda)}. \qquad (2)$$

Aim:

Estimate integral (2) in the critical region $K(\exp B(2t\rho, r(t))K$.

Critical region: Spherical analysis

Follow idea of Anker-Ji: Harish–Chandra expansion of spherical functions away from the walls,

$$\varphi_\lambda(\exp H) = \sum_{w\in W} \mathbf{c}(w.\lambda)\,\Phi_{w.\lambda}(H),$$

where

$$\Phi_\lambda(\exp H) = e^{\langle i\lambda-\rho, H\rangle} \sum_{q\in 2Q} \gamma_q(\lambda)\, e^{-\langle q, H\rangle},$$

where Q: positive root lattice.

Critical region: Spherical analysis

Follow idea of Anker-Ji: Harish–Chandra expansion of spherical functions away from the walls,

$$\varphi_\lambda(\exp H) = \sum_{w \in W} \mathbf{c}(w.\lambda)\, \Phi_{w.\lambda}(H),$$

where

$$\Phi_\lambda(\exp H) = e^{\langle i\lambda - \rho, H\rangle} \sum_{q \in 2Q} \gamma_q(\lambda)\, e^{-\langle q, H\rangle},$$

where Q: positive root lattice.

Replace into $u(t, \exp H) - Mh_t(\exp H) = \int_{\mathfrak{a}} \dots$

Critical region: Spherical analysis

Contribution of $\Omega(\lambda) = \mathcal{H}f(\lambda) - \mathcal{H}f(i\rho)$:

- Compact support of initial data $f \rightsquigarrow$
  Paley-Wiener: $\mathcal{H}f$ holomorphic in $\mathfrak{a}_{\mathbb{C}} \rightsquigarrow \Omega(\lambda)$ holomorphic.
- All other terms in the integral $\int_{\mathfrak{a}} \dots$ holomorphic in $\mathfrak{a} + i\overline{\mathfrak{a}^+}$.
- Change of contour $\lambda \longmapsto \lambda + i\frac{H}{2t}$ and rescale $\lambda \longmapsto \frac{\lambda}{\sqrt{t}}$.

Critical region: Spherical analysis

- Paley-Wiener (again) for $f \in C_c^\infty(K\backslash G/K)$:

$$\exists A, \forall \xi \in \mathfrak{a}_{\mathbb{C}}, \forall j, N \in \mathbb{N} : \ |\nabla_\xi^j \mathcal{H}f(\xi)| \leq C\,(1 + |\xi|)^{-N} e^{A|\operatorname{Im}\xi|}. \tag{3}$$

- Control $\Omega\big(\frac{\lambda}{\sqrt{t}} + i\,\frac{H}{2t}\big) = \mathcal{H}f\big(\frac{\lambda}{\sqrt{t}} + i\,\frac{H}{2t}\big) - \mathcal{H}f(i\rho)$ by (3):

$$\Big|\Omega\big(\tfrac{\lambda}{\sqrt{t}} + i\,\tfrac{H}{2t}\big)\Big| \lesssim \underbrace{e^{\text{const.}\frac{|H|}{2t}}}_{\text{bdd}}\Big(\frac{|\lambda|}{\sqrt{t}} + \underbrace{\Big|\frac{H}{2t} - \rho\Big|}_{\lesssim \frac{r(t)}{t}}\Big)$$

in critical region where $|H - 2t\rho| \leq r(t)$, $\frac{r(t)}{\sqrt{t}} \to +\infty$, $\frac{r(t)}{t} \to 0$.

Critical region: Spherical analysis

Eventually

$$|u(t,x) - M\,h_t(x)| \lesssim \underbrace{\frac{r(t)}{t}}_{\text{extra decay in time!}} \times \{\text{smth comparable to heat kernel}\}$$

So,

$$\int_{K(\exp B(2t\rho, r(t))K} dx\, |u(t,x) - Mh_t(x)|$$

tends to 0 at speed $\frac{r(t)}{t}$.

Distinguished Laplacian

Iwasawa dec. $G = N(\exp \mathfrak{a})K \rightsquigarrow G/K$ can be identified w. solvable Lie group $S = N(\exp \mathfrak{a}) = (\exp \mathfrak{a})N$.

Distinguished Laplacian

Iwasawa dec. $G = N(\exp \mathfrak{a})K \rightsquigarrow G/K$ can be identified w. solvable Lie group $S = N(\exp \mathfrak{a}) = (\exp \mathfrak{a})N$. Modular function:

$$\widetilde{\delta}(n(\exp H)) = \widetilde{\delta}((\exp H)n) = e^{-2\langle \rho, H \rangle}, \quad H \in \mathfrak{a}.$$

Distinguished Laplacian

$$\widetilde{\Delta} = \widetilde{\delta}^{\frac{1}{2}} \circ (\Delta + |\rho|^2) \circ \widetilde{\delta}^{-\frac{1}{2}}$$

on S, self-adjoint with respect to right-inv. Haar measure $d_r g$.

Distinguished Laplacian

Heat kernel:

$$\widetilde{h}_t(g) = \widetilde{\delta}(g)^{\frac{1}{2}} \, e^{|\rho|^2 t} \, h_t(g).$$

Lemma

Let $t \mapsto r(t)$ be positive increasing function s.t. $\frac{r(t)}{\sqrt{t}} \to \infty$. Then

$$\lim_{t \to +\infty} \int_{g \in S \text{ s.t. } |g| \geq r(t)} d_r g \, \widetilde{h}_t(g) = 0,$$

i.e., heat kernel $\widetilde{h}_t$ on S concentrates asympt. in ball $B(e, r(t))$, as in euclidean case.

Distinguished Laplacian

Let f be K-bi-invariant, $\widetilde{f}(g) := \widetilde{\delta}^{1/2}(g)f(g)$, and

$$\widetilde{M} = \int_S d_r g \widetilde{f}(g) = \int_G dg \varphi_0(g) f(g).$$

Distinguished Laplacian

Let f be K-bi-invariant, $\widetilde{f}(g) := \widetilde{\delta}^{1/2}(g)f(g)$, and

$$\widetilde{M} = \int_S d_r g \widetilde{f}(g) = \int_G dg \varphi_0(g) f(g).$$

Theorem

Let $\widetilde{f}$ as above, smooth, comp. supported. If $\widetilde{u}(t,g) = (\widetilde{f} * \widetilde{h}_t)(g)$ is the sol. to

$$\begin{cases} \partial_t \widetilde{u}(t,g) = \widetilde{\Delta}_g \widetilde{u}(t,g) \\ \widetilde{u}(0,g) = \widetilde{f}(g), \end{cases}$$

then

$$\lim_{t\to+\infty} \int_S d_r g \, |\widetilde{u}(t,g) - \widetilde{M}\widetilde{h}_t(g)| = 0.$$

Non radial solutions

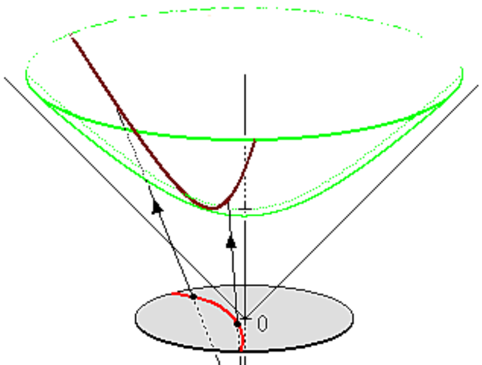
The asymptotic L^1 result may fail for general non K-bi-invariant solutions to heat eq. Candidate: a displaced heat kernel.

Example

$$\mathbb{H}^3 = \{Y = (x_0, x_1, x_2, x_3) \in \mathbb{R}^4,\ x_0^2 - x_1^2 - x_2^2 - x_3^2 = 1,\ x_0 > 0\}.$$

or in polar coordinates

$$Y = (\cosh r, \sinh r\, \omega), \quad r > 0,\ \omega \in \mathbb{S}^2.$$



Non radial solutions

Example

Distance to the origin $O = (1, 0, 0, 0)$:

$$d(Y, O) = d((\cosh r, \sinh r\ \omega), O) = r,$$
$$d(Y', O) = d((\cosh r', \sinh r'\ \omega'), O) = r'.$$

Distance between arbitrary points Y, Y':

$$d(Y, Y') = \cosh^{-1}(\cosh r \cosh r' - \sinh r \sinh r'\ \omega \cdot \omega') := d$$

Nonradial solutions

Example

Fix $Y' \neq$ origin $\rightsquigarrow$

$h_t(d(Y,O)) = h_t(r)$, $h_t(d(Y,Y')) = h_t(d)$ integr. sol. of heat eq.

Nonradial solutions

Example

Fix $Y' \neq$ origin $\rightsquigarrow$
$h_t(d(Y,O)) = h_t(r)$, $h_t(d(Y,Y')) = h_t(d)$ integr. sol. of heat eq.

By explicit expression of heat kernel in $\mathbb{H}^3$,

$$\frac{h_t(d(Y,Y'))}{h_t(d(Y,O))} = \frac{d}{r}\frac{\sinh r}{\sinh d} e^{-\frac{r}{4t}(d-r)(1+\frac{d}{r})}.$$

In critical region: $r \sim 2t$.

Nonradial solutions

Example

$$\| \underbrace{h_t(d(\cdot, Y')) - h_t(d(\cdot, O))}_{h_t(d(\cdot,O))\left(\frac{h_t(d(\cdot,Y'))}{h_t(d(\cdot,O))}-1\right)} \|_{L^1\{\text{critical region}\}} \geq$$

$$\|h_t\|_{L^1\{\text{critical region}\}} \int_{\mathbb{S}^2} \left| f(r', \omega \cdot \omega', t) - 1 \right| d\omega,$$

where the integral does not vanish as $t \to \infty$.

Summary

- $X = \mathbb{R}^n$, $\|u - Mh_t\|_{L^1} \to 0$, for all initial data $f \in L^1(\mathbb{R}^n)$.
- $X =$ symm. space of noncompact type, $\|u - Mh_t\|_{L^1} \to 0$, for all initial data $f \in L^1(K \backslash G / K)$.
- Work in progress: distinguished Laplacian, non K-bi-invariant solutions, Doob Laplacian, convergence in other L^p norms...

Thank you for your attention!