# On vertex-algebraic proof of complete reducibility of certain categories of modules for affine Lie algebras

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Znanstveni centar izvrsnosti
za kvantne i kompleksne sustave te
reprezentacije Liejevih algebri

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# Complete reducibility in Representation Theory

- $\mathfrak{g}$ simple Lie algebra over a field of char. zero, every finite-dimensional $\mathfrak{g}$–module is semi-simple [Weyl theorem of complete reducibility]
- It does not hold for Lie superalgebras except $\mathfrak{g} = osp(1, 2n)$.

  $\widehat{\mathfrak{g}}$–affine Lie algebra.
- The category of $\widehat{\mathfrak{g}}$–integrable modules in the category $\mathcal{O}$ is semi-simple
- The Kazhdan-Lusztig category $KL_k$ of $\widehat{\mathfrak{g}}$–modules is semi-simple for $k$-generic
- If $V$ is a rational vertex operator algebra, then the category of $V$–modules is semi-simple.
- The Zhu's algebra of a rational vertex operator algebra is semi-simple.

# Methods of proving complete reducibility for affine vertex algebras

- Lie theoretic approach, by proving that $Ext^1(M, N) = \{0\}$ in certain categories: Kac-Wakimoto[1988], Kac-Gorelik[2007]
- Vertex algebraic methods which use certain aspects of represenetation theory of VOAs: [Adamović-Kac-Moseneder-Papi-Perše, IMRN 2020], [Arakawa, 2016]
- Tensor category (TC) approach, recent papers by Creutzig, McRea, Yang, using tensor product theory of VOA modules.

# Definition of vertex algebra

Vertex algebra is a triple $(V, Y, \mathbf{1})$ where
$V$ complex vector space; $\mathbf{1}$ *vacuum* vector, $Y$ is a linear map

$$\begin{aligned} Y(\cdot, z): \qquad & V \to (\text{End } V)[[z, z^{-1}]]; \\ & a \mapsto Y(a, z) = \sum_{n \in \mathbb{Z}} a_n z^{-n-1} \in (\text{End } V)[[z, z^{-1}]] \end{aligned}$$

which satisfies the following conditions on $a, b \in V$:

# Definition of a vertex algebra

$a_n b = 0$ for $n$ sufficiently large.

$[D, Y(a,z)] = Y(D(a),z) = \frac{d}{dz} Y(a,z)$,

where $D \in \text{End } V$ is defined by $D(a) = a_{-2}\mathbf{1}$.

$Y(\mathbf{1}, z) = I_V$.

$Y(a,z)\mathbf{1} \in V[[z]]$ and $\lim_{z \to 0} Y(a,z)\mathbf{1} = a$.

There exist $N \geq 0$ (which depends on $a$ and $b$) such that

$$(z_1 - z_2)^N [Y(a, z_1), Y(b, z_2)] = 0 \quad \text{(locality)}.$$

# Representations of vertex algebras

Representation (module) for vertex algebra $V$ is a pair $(M, Y_M)$ where

$M$ is a complex vector space , and $Y_M(\cdot, z)$ is a linear map

$$Y_M : V \to \text{End}(M)[[z, z^{-1}]], \ a \mapsto Y_M(a, z) = \sum_{n \in \mathbb{Z}} a_n z^{-n-1},$$

which satisfies certain axioms....

# Virasoro vectors and conformal embeddings

- Let $Vir = \oplus_{n\in\mathbb{Z}}\mathbb{C}L(n) \oplus \mathbb{C}C$ be a Virasoro algebra, i.e.,
- $[L(m), L(n)] = (m-n)L(m+n) + \frac{m^3-m}{12}\delta_{m+n,0}C$
- $C$ is central element.
- Vector $\omega$ in vertex algebra $V$ is called conformal (or Virasoro vector) of central charge $c$ if components of the field

$$Y(\omega, z) = \sum_{n\in\mathbb{Z}} L(n)z^{-n-2}$$

  define a representation of the Virasoro algebra of central charge $c$.
- Let $V$ be a vertex algebra with conformal vector $\omega_V$, $U$ be its subalgebra with conformal vector $\omega_U$. $U$ is conformally embedded into $V$ if

$$\omega_U = \omega_V.$$

# Rational vertex algebras

- A vertex algebra $V$ is called **rational** if it has finitely many irreducible modules and if the category of $V$–modules is semisimple.
- Rational vertex algebras correspond to **rational conformal field theory**

  Examples:

  Vertex algebras associated to integrable representations of affine Kac–Moody Lie algebras

  Minimal models for Virasoro algebras, superconformal algebras, W-algebras

# Affine Lie superalgebras

Let $\mathfrak{g}$ be a finite-dimensional simple Lie superalgebra over $\mathbb{C}$ and let $(\cdot,\cdot)$ be a nondegenerate super-symmetric bilinear form on $\mathfrak{g}$.

The affine Lie superalgebra $\hat{\mathfrak{g}}$ associated with $\mathfrak{g}$ is defined as

$$\hat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}K$$

where $K$ is the canonical central element and the Lie algebra structure is given by

$$[x \otimes t^n, y \otimes t^m] = [x, y] \otimes t^{n+m} + n(x, y)\delta_{n+m,0}K.$$

We will say that $M$ is a $\hat{\mathfrak{g}}$–module of level $k$ if the central element $K$ acts on $M$ as a multiplication with $k$.

# Affine vertex algebras

- $V^k(\mathfrak{g})$ universal affine vertex algebra of level $k$ ($k \neq -h^\vee$).
- As $\hat{\mathfrak{g}}$–module $V^k(\mathfrak{g}) = U(\hat{\mathfrak{g}}) \otimes_{U(\hat{\mathfrak{g}}_{\geq 0} + \mathbb{C}K)} 1$.
- $V_k(\mathfrak{g})$ simple quotient of $V^k(\mathfrak{g})$.
- Let $x_i, y_i$, $i = 1, \dots, \dim \mathfrak{g}$ be dual bases of $\mathfrak{g}$ with respect to form $(\cdot, \cdot)$, and let

$$\omega_{sug} = \frac{1}{2(k + h^\vee)} \sum_{i=1}^{\dim \mathfrak{g}} x_i(-1) y_i(-1) \mathbf{1} \in V_k(\mathfrak{g}).$$

- $\omega_{sug}$ Sugawara Virasoro vector in $V_k(\mathfrak{g})$ of central charge

$$c(sug) = \frac{k \mathrm{sdim} \mathfrak{g}}{k + h^\vee}.$$

# Notations, terminology

- Let $KL^k$ be the subcategory of $\mathcal{O}^k$ consisting of modules $M$ on which $\mathfrak{g}$–acts locally finite.
- Modules from $KL^k$ are $V^k(\mathfrak{g})$–modules.
- Category $\mathcal{O}_k$: $V_k(\mathfrak{g})$–modules which are in $\mathcal{O}^k$.
- Category $KL_k$: $V_k(\mathfrak{g})$–modules which are in $KL^k$.
- Important problem: Classify irreducible modules in $KL_k$.
- For generic $k$: $KL^k = KL_k$
- Classified for $k$ admissible by T. Arakawa (2015) (conjectured by D.A, A.Milas in 1995)
- $\mathcal{O}_k$ is semi-simple for $k$-admissible.

# Semi-simplicity at admissible levels

- Kac Wakimoto in 1988 define the notion of admissible levels and admissible highest weight modules
- Ex. $\mathfrak{g} = \mathfrak{sl}(n)$. Level $k$ is called admissible if $k + n = \frac{p'}{p}$ such that $p, p' \in \mathbb{Z}_{>0}$, $(p, p') = 1$ and $p' \geq n$.
- For two admissible irreducible highest weight module $M, N$ of level $k$, Kac-Wakimoto proved (using Lie-theoretic proof):

$$\mathrm{Ext}^1(M, N) = \{0\}.$$

- VOA classification says that every irreducible modules at admissible level $k$ is admissible.
- This proves that $\mathcal{O}_k$ is semi-simple on admissible level.

# Affine $W$ algebra $W^k(\mathfrak{g}, f_\theta)$

- Choose root vectors $e_\theta$ and $f_\theta$ such that

$$[e_\theta, f_\theta] = x, \ [x, e_\theta] = e_\theta, \ [x, f_\theta] = -f_\theta.$$

- ad$(x)$ defines minimal $\frac{1}{2}\mathbb{Z}$–gradation:

$$\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_{-1/2} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_{1/2} \oplus \mathfrak{g}_1.$$

- Let $\mathfrak{g}^\natural = \{a \in \mathfrak{g}_0 \mid (a|x) = 0\}$.
- $W^k(\mathfrak{g}, f_\theta)$ is strongly generated by vectors
- $G^{\{u\}}$, $u \in \mathfrak{g}_{-1/2}$, of conformal weight $3/2$;
- $J^{\{a\}}$, $u \in \mathfrak{g}^\natural$ of conformal weight 1;
- $\omega$ conformal vector of central charge

$$c(\mathfrak{g}, k) = \frac{k \mathrm{sdim} \mathfrak{g}}{k + h^\vee} - 6k + h^\vee - 4.$$

- $W_k(\mathfrak{g}, f_\theta)$ simple quotient of $W^k(\mathfrak{g}, f_\theta)$

# Collapsing levels

- If $W_k(\mathfrak{g}, f_\theta) = \mathcal{V}_k(\mathfrak{g}^\natural)$, we say that $k$ is a collapsing level.
- If $k$ is a collapsing level and if $\mathcal{V}_k(\mathfrak{g}^\natural)$ is not affine vertex algebra at the critical level, then $k$ is a conformal level.
- In [AKMPP, J. Algebra (2018)] we classified all collapsing levels.

- Interesting cases of collapsing levels:

1 $\mathcal{V}_k(\mathfrak{g}^\natural) = \mathbb{C}\mathbf{1}$ .
2 $\mathcal{V}_k(\mathfrak{g}^\natural)$ is a rational vertex algebra.
3 $\mathcal{V}_k(\mathfrak{g}^\natural)$ is admissible affine vertex algebra.
4 $\mathcal{V}_k(\mathfrak{g}^\natural)$ is an affine vertex algebra at the critical level.

# Collapsing levels

The following theorem was proved by Arakawa–Moreau (2018): Lie algebra case, AKMPP, J. Algebra (2018): Lie superalgebra case.

**Theorem**

*$W_k(\mathfrak{g}, f_\theta) = \mathbb{C}\mathbf{1}$ iff we are in one of the following cases*

(1) $k = -\frac{h^\vee}{6} - 1$ *and* $\mathfrak{g}$ *is one of the Lie algebras of exceptional Deligne's series:* $A_2$, $G_2$, $D_4$, $F_4$, $E_6$, $E_7$, $E_8$, *or* $\mathfrak{g} = psl(m|m)$ $(m \geq 2)$, $osp(n+8|n)$ $(n \geq 2)$, $spo(2|1)$, $F(4)$, $G(3)$

(2) $k = -1/2$, $\mathfrak{g} = spo(n|m)$, $n \geq 1$.

# Identification of rational VOAs

In the following cases of collapsing levels we identify a rational VOA:

- $W_{-2}(osp(m|n), f_\theta) = V_{\frac{m-n}{4}-2}(sl(2))$ is rational for $m-n \in 2\mathbb{Z}$, $m-n \geq 8$.
- $W_{-4/3}(G_2, f_\theta) = V_1(sl(2))$;
- $W_{-3/4}(spo(2|3), f_\theta) = V_1(sl(2))$.
- $W_{-\frac{n+1}{n+2}}(D(2,1;-\frac{n+1}{n+2}), f_\theta) = V_n(sl(2))$, $n \in \mathbb{Z}_{\geq 1}$.

# Semi-simplicity in $KL_k$

We prove the following result on complete reducibility result in $KL_k$

**Theorem (AKMPP, 2020)**

*Assume that $\mathfrak{g}$ is a Lie algebra and $k \in \mathbb{C} \setminus \mathbb{Z}_{\geq 0}$. Then $KL_k$ is a semi-simple category in the following cases:*

- *$k$ is a collapsing level.*
- *$W_k(\mathfrak{g}, \theta)$ is a rational vertex operator algebra.*

# Semi-simplicity in $KL_k$: Examples

The category $KL_k$ is semi-simple in the following cases:

- $\mathfrak{g} = D_{\ell+1}$, $\mathfrak{g} = B_\ell$, $\ell \geq 2$, $k = -2$,
- $\mathfrak{g} = A_\ell$, $\ell \geq 3$, $k = -1$, $\mathfrak{g} = A_{2\ell-1}$, $\ell \geq 2$, $k = -\ell$.
- $\mathfrak{g} = D_{2\ell}$, $\ell \geq 3$, $k = -2\ell + 3$,
- $g = E_6$, $k = -4$, $\mathfrak{g} = E_7$, $k = -6$,
- $\mathfrak{g} = C_\ell$, $k = -1 - \ell/2$,
- $\mathfrak{g} = F_4$, $k = -3$.

# On the proof of completely reducibility

- Collapsing levels from previous theorem are not admissible, and in the category $\mathcal{O}_k$ we do have indecomposable modules and non-trivial extensions between irreducibles.
- We need to prove that $Ext^1(M, N) = \{0\}$ in the category $KL_k$.

(C) The proof is reduced for proving that for any h.w. module $M$ in $KL_k$, $M$ is irreducible.

- We use QHR function $H_{f_\theta}$ which is exact and non-zero in $KL_k$ for our collapsing levels. Then

$$H_{f_\theta}(V_k(\mathfrak{g})) = \mathcal{W}_k(\mathfrak{g}, \theta).$$

  and for any module $M$ in $KL_k$, we have that $H_{f_\theta}(M)$ is a $\mathcal{W}_k(\mathfrak{g}, \theta)$–module.
- Using RT of $\mathcal{W}_k(\mathfrak{g}, \theta)$, and properties of functor $H_{f_\theta}$, we check (C), implying complete-reducibility.

# VOAs with exactly one ordinary module

**Theorem (AKMPP, IMRN, 2020)**

*Assume that level $k$ and the basic simple Lie superalgebra $\mathfrak{g}$ satisfy one of the following conditions:*

(1) $k = -\frac{h^\vee}{6} - 1$ *and $\mathfrak{g}$ is one of the Lie algebras of exceptional Deligne's series* $A_2$, $G_2$, $D_4$, $F_4$, $E_6$, $E_7$, $E_8$, *or* $\mathfrak{g} = psl(m|m)$ $(m \geq 2)$, $osp(n+8|n)$ $(n \geq 2)$, $spo(2|1)$, $F(4)$, $G(3)$ *(for both choices of $\theta$);*

(2) $k = -h^\vee/2 + 1$ *and* $\mathfrak{g} = osp(n+4m+8, n)$, $n \geq 2, m \geq 0$.

(3) $k = -h^\vee/2 + 1$ *and* $\mathfrak{g} = D_{2m}$, $m \geq 2$.

*Then $V_k(\mathfrak{g})$ is the unique irreducible $V_k(\mathfrak{g})$–module in the category of ordinary $V_k(\mathfrak{g})$–modules.*

# The case of other nilpotent elements

- For any nilpotent element $f$ of $\mathfrak{g}$, one defines universal $\mathcal{W}$–algebras $\mathcal{W}^k(\mathfrak{g}, f)$ as $H_f(V^k(\mathfrak{g}))$. Consider simple quotient $\mathcal{W}_k(\mathfrak{g}, f)$.
- Level $k$ is called collapsing if $\mathcal{W}_k(\mathfrak{g}, f)$ collapses to its affine vertex subalgebra.
- In the case of admissible affine vertex algebras, such collapsing levels are investigated by Arakawa-van Ekeren-Moreau (2021). But at admissible levels we already know that $V_k(\mathfrak{g})$ is semi-simple.
- Question is what is happening beyond admissible levels?
- We have two problems:

(1) Construct and classify collapsing non-admissble $V_k(\mathfrak{g})$.

(2) Check condition (C) for modules in $KL_k$

# The case of other nilpotent elements

- It is natural to use QHR $H_f(\cdot)$.
- In the case of minimal reduction $f = f_\theta$, we know a priori that $H_f(M)$ is non-zero for any module in $KL_k$. Moreover, we could classify modules in $KL_k$ using properties of $H_f(M)$.
- Unfortunately, such results in unknown in general. The key property is to investigate vanishing and non-vanishing of $H_f(M)$.
- Quite surprisingly, we indeed have modules $M$ in $KL_k$ such that $H_f(M) = \{0\}$. So methods of [AKMPP, IMRN, 2020] can not be directly applied.
- Our method is (still very much work in progress in general):

(1) Classify irreducible $V_k(\mathfrak{g})$–modules without using results of (non)vanishing of cohomology.

(2) Check directly (a posteriori to classification) when $H_f(M)$ is zero or non-zero, for projective covers of irreducible modules.

(3) Prove property (C) using (1), (2) and some symmetries of VOAs.

# The case $k = -5/2$ and $\mathfrak{g} = sl(4)$

In a joint work with O. Perše, I. Vukorepa, we test this strategy:

- We classify irreducible modules by hard calculations.
- We show that $k = -5/2$ is a collapsing level for $f = f_{suberg}$.
- We get a family of irreducible modules $M_n$, $n \in \mathbb{Z}$ such that

$$H_f(M_n) \neq 0 \quad (n \geq 0), \quad H_f(M_n) = 0 \quad (n < 0).$$

- Using this and applying a VOA automorphism, we check the condition (C) and prove that $KL_k$ is a semi-simple.
- Moreover, $KL_k$ is a rigid, braided tensor category.

# A relation with vertex tensor categories and conformal embeddings

- Recent results in the VOA theory by R. McRea and collaborators say: Assume that there is a conformal embeddings $V \hookrightarrow W$ of VOAs $V$ and $W$ such that the the category $\mathcal{C}$ of $V$–modules admits the vertex algebraic braided tensor category (TC) structure of Huang-Lepowsky-Zhang. Then the simplicity of the category of $W$–modules, implies the simplicity of $\mathcal{C}$.
- Assume that we have conformal embeddings $V_k(\mathfrak{g}_0) \hookrightarrow V_k(\mathfrak{g})$, and $V_k(\mathfrak{g})$ is admissible vertex algebra.
- Under condition that the category $KL_k$ for $V_k(\mathfrak{g}_0)$ is a braided vertex tensor category, then $KL_k$ is semi-simple.
- Problem is that we don't know a priori that $V_k(\mathfrak{g}_0)$ has the braided vertex tensor category structure. But it is expected that this conjecture can be proved in general.

# A relation with vertex tensor categories and conformal embeddings

- Series of joint papers with Kac, Moseneder-Frajria, Papi, Perše give a family of examples for which we can apply previous arguments:
- $V_k(gl(2n)) = V_k(sl(2n)) \otimes M(1) \hookrightarrow V_k(sl(2n+1))$ at $k = -\frac{2n+1}{2}$, for $n \geq 2$.
- Since $V_k(sl(2n+1))$ is admissible, we expect that

  $KL_k$ for $V_k(sl(2n))$ is a semi-simple for each $n \geq 2$.

  Proved for $n = 2$ in [APV, 2021].

  Using conformal embedding [AP, SIGMA, 2012]

  $$V_{-1}(C_n) \hookrightarrow V_{-1}(sl(2n)),$$

  we expect:

  $KL_k$ is semi-simple for $L_{-1}(C_n)$ and $n \geq 2$.

# Thank you!