Towards quantum fields on causal symmetric spaces (jt. with Gestur Ólafsson, Vincenzo Morinelli)

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Outline

Background on von Neumann algebras

$\mathcal{H}$ a complex Hilbert space, $B(\mathcal{H})$ bounded operators on $\mathcal{H}$

Commutant of $\mathcal{M} \subseteq B(\mathcal{H})$: $\mathcal{M}' = \{a \in B(\mathcal{H}) \colon (\forall s \in \mathcal{M}) as = sa\}$

von Neumann algebra: $\mathcal{M} \subseteq B(\mathcal{H})$ a $*$-subalgebra with $\mathcal{M} = \mathcal{M}''$.

For a von Neumann algebra $\mathcal{M} \subseteq B(\mathcal{H})$, a vector $\Omega \in \mathcal{H}$ is called

- cyclic if $\overline{\mathcal{M}\Omega} = \mathcal{H}$.
- separating if $M \in \mathcal{M}, M\Omega = 0$ implies $M = 0$.

Theorem (Tomita 1967, Takesaki 1970)

Any cyclic and separating vector $\Omega \in \mathcal{H}$ for the von Neumann algebra $\mathcal{M}$ determines a conjugation J (=antilinear isometry) and a positive selfadjoint operator $\Delta > 0$ (modular operator) such that

(i) $J\Delta J = \Delta^{-1}$ *(modular relation).*

(ii) $J\mathcal{M}J = \mathcal{M}'$*; in particular* $\mathcal{M}' \cong \mathcal{M}^{\mathrm{op}}$ *(via $a \mapsto Ja^*J$).*

(iii) $\Delta^{it}\mathcal{M}\Delta^{-it} = \mathcal{M}$ *for every $t \in \mathbb{R}$ (modular automorphism group)*

From Ω to (Δ, J): The operator $S(M\Omega) = M^*\Omega$, $M \in \mathcal{M}$, is densely defined. Its closure has the polar decomposition $\overline{S} = J\Delta^{1/2}$.

Von Neumann algebras in Quantum Field Theory (QFT)

In QFT one studies nets of von Neumann algebras $(\mathcal{M}(\mathcal{O}))_{\mathcal{O}\subseteq M}$ in $B(\mathcal{H})$. Here $\mathcal{M}(\mathcal{O})$ corresponds to observables measurable in the "laboratory" $\mathcal{O} \subseteq M$, an open subset of the space-time manifold M.

Requirements:

- $\mathcal{O}_1 \subseteq \mathcal{O}_2$ implies $\mathcal{M}(\mathcal{O}_1) \subseteq \mathcal{M}(\mathcal{O}_2)$ (Isotony)
- $\mathcal{O}_1 \subseteq \mathcal{O}_2'$ implies $\mathcal{M}(\mathcal{O}_1) \subseteq \mathcal{M}(\mathcal{O}_2)'$ (Locality)
  ($\mathcal{O}' =$ causal compl. $=$ events that cannot exchange signals with $\mathcal{O}$)
- There is a unitary representation $(U, \mathcal{H})$ of a space time symmetry group G such that, for $g \in G$,
  $U(g)\mathcal{M}(\mathcal{O})U(g)^{-1} = \mathcal{M}(g\mathcal{O})$ (Covariance)

Given a unit vector $\Omega \in \mathcal{H}$ (vacuum state), we call $\mathcal{O} \subseteq M$ a test region if Ω is cyclic and separating for $\mathcal{M}(\mathcal{O})$.
The Tomita-Takesaki-Thm. provides $(\Delta_{\mathcal{O}}, J_{\mathcal{O}})$ and
$\Delta_{\mathcal{O}}^{-it/2\pi}$ defines dynamics on $\mathcal{M}(\mathcal{O})$ (flow of time, Connes-Rovelli '94).
How do we get such structures?

Second quantization algebras (bosonic case)

For a complex Hilbert space $\mathcal{H}$, consider the bosonic Fock space

$$\mathcal{F}_+(\mathcal{H}) = \widehat{\bigoplus}_{n=0}^\infty S^n(\mathcal{H}) \quad \text{with} \quad S^0(\mathcal{H}) = \mathbb{C}\Omega, \quad S^1(\mathcal{H}) = \mathcal{H},$$

and the annihilation and creation operators

$$a(\xi)\Omega = 0, \quad a(\xi)(\xi_1 \vee \cdots \vee \xi_n) = \sum_{j=1}^n \langle \xi, \xi_j \rangle \xi_1 \vee \cdots \vee \widehat{\xi_j} \vee \cdots \vee \xi_n$$

$$a^*(\xi)(\xi_1 \vee \cdots \vee \xi_n) = \xi \vee \xi_1 \vee \cdots \vee \xi_n$$

satisfying the canonical commutation relations (CCR)

$$[a(f), a^*(g)] = \langle f, g \rangle \mathbf{1}, \quad [a(f), a(g)] = [a^*(f), a^*(g)] = 0.$$

They lead to the unitary Weyl operators $W(v) = e^{\frac{i}{\sqrt{2}}\overline{a^*(v)+a(v)}}$.
They satisfy the Weyl relations

$$W(v)W(w) = e^{-i\,\mathrm{Im}\langle v,w\rangle/2} W(v+w), \qquad v, w \in \mathcal{H}$$

and define an irreducible rep. of the Heisenberg group $\mathrm{Heis}(\mathcal{H})$ on $\mathcal{F}_+(\mathcal{H})$.

For each closed real subspace $V \subseteq \mathcal{H}$, we obtain the von Neumann algebra

$$\mathcal{R}(V) := W(V)'' = \{W(v) \colon v \in V\}'' \subseteq B(\mathcal{F}_+(\mathcal{H})).$$

Properties (Araki, 1960s):

- Isotony: $\mathcal{R}(V) \subseteq \mathcal{R}(W)$ iff $V \subseteq W$ ($\Rightarrow \mathcal{R}$ injective).
- Duality: $\mathcal{R}(V)' = \mathcal{R}(V')$ for $V' := \{x \in \mathcal{H} \colon \mathrm{Im}\langle x, V\rangle = \{0\}\}$
  symplectic complement $\leftrightarrow$ commutant $\leftrightarrow$ causal complement
- Covariance: $\mathcal{R}(gV) = U(g)\mathcal{R}(V)U(g)^{-1}$ for $g \in \mathrm{U}(\mathcal{H})$,
  U Fock rep. of $\mathrm{U}(\mathcal{H})$: $U(g)(\xi_1 \vee \cdots \vee \xi_n) = g\xi_1 \vee \cdots \vee g\xi_n$.

- $\mathcal{R}(\mathcal{H}) = B(\mathcal{F}_+(\mathcal{H}))$ (irred. of rep. of $\mathrm{Heis}(\mathcal{H})$ and Schur Lemma).
- V is cyclic: $\overline{V + iV} = \mathcal{H}$ iff $\Omega \in \mathcal{F}_+(\mathcal{H})$ is cyclic for $\mathcal{R}(V)$.
- V is separating: $V \cap iV = \{0\}$ iff $\Omega \in \mathcal{F}_+(\mathcal{H})$ is separating for $\mathcal{R}(V)$.
- V is **standard** (cyclic and separating) iff $\Omega \in \mathcal{F}_+(\mathcal{H})$ is cyclic and separating for $\mathcal{R}(V)$.
  Then $S(v + iw) := v - iw$ on $V + iV$ is a closed operator on $\mathcal{H}$, polar decomposition $S = J_V \Delta_V^{1/2}$ defines a conjugation and a positive operator. **(Modular structure lives on standard subspaces of $\mathcal{H}$)**.

Nets of real subspaces on homogeneous spaces

Let G be a Lie group, $P \subseteq G$ a closed subgroup
$M = G/P$ a homogeneous space and
$(U, \mathcal{H})$ a unitary representation of G.

Definition

A net of real subspaces on M is a family $V(\mathcal{O}) \subseteq \mathcal{H}$ of closed real subspaces, $\mathcal{O} \subseteq M$ open, such that

- $\mathcal{O}_1 \subseteq \mathcal{O}_2$ implies $V(\mathcal{O}_1) \subseteq V(\mathcal{O}_2)$ (Isotony)
- $U(g)V(\mathcal{O}) = V(g\mathcal{O})$ (Covariance)

Then second quantization leads to the net $\mathcal{R}(V(\mathcal{O}))$ of von Neumann algebras on the Fock space $\mathcal{F}_+(\mathcal{H})$ satisfying isotony and covariance.

Problems:

- How to construct such nets containing standard subspaces?
- Which domains $\mathcal{O} \subseteq M$ correspond to standard subspaces $V(\mathcal{O})$?
- Geometric modular group: $\Delta_{V(\mathcal{O})}^{-it/2\pi} = U(\exp th)$ for some $h \in \mathfrak{g}$?

Antiunitary representations

Def: A graded group is a pair (G, G_+), where $G_+ \subseteq G$ of index 2; $G_- := G \setminus G_+$.

Examples: (of graded groups, where G_+ is the identity component):

- $\mathbb{R}^\times \cong \mathbb{R} \times \mathbb{Z}_2$ (dilation group)
- Antiunitary group $\mathrm{AU}(\mathcal{H})$ of all unitary and antiunitary operators on $\mathcal{H}$ with $\mathrm{AU}(\mathcal{H})_+ = \mathrm{U}(\mathcal{H})$.
- Projective (Möbius) group $\mathrm{PGL}_2(\mathbb{R})$, acting on $\mathbb{P}_1(\mathbb{R}) \cong \mathbb{S}^1$
- Proper Poincaré group $\mathbb{R}^d \rtimes \mathrm{SO}_{1,d-1}(\mathbb{R})$, acting on $\mathbb{R}^{1,d-1}$.

Typical structure: $G \cong G_+ \rtimes \{\mathrm{id}_G, \tau_G\}$, τ_G involutive autom. of G_+.

Def: An antiunitary representation $(U, \mathcal{H})$ of a graded Lie group (G, G_+) is a morphism $U : G \to \mathrm{AU}(\mathcal{H})$ of graded groups, i.e., $G_+ = U^{-1}(\mathrm{U}(\mathcal{H}))$, for which all orbit maps $U^\xi : G \to \mathcal{H}, g \mapsto U_g\xi$ are continuous.

Brunetti–Guido–Longo construction

Let $V \subseteq \mathcal{H}$ standard ($V + iV$ dense and $V \cap iV = \{0\}$).
Then $V = \mathrm{Fix}(J_V \Delta_V^{1/2})$, J_V conjugation, $\Delta_V > 0$, $J_V \Delta_V J_V = \Delta_V^{-1}$.

Encoding std subspaces in representations: For $V \subseteq \mathcal{H}$ standard,

$$U^V \colon \mathbb{R}^\times \to \mathrm{AU}(\mathcal{H}), \quad U^V(e^t) := \Delta_V^{-it/2\pi}, \quad U^V(-1) := J_V$$

is an antiunitary representation of the graded group $\mathbb{R}^\times$. We thus obtain a **bijection**

$$\mathrm{Stand}(\mathcal{H}) \to \underbrace{\mathrm{Hom}_{\mathrm{gr}}(\mathbb{R}^\times, \mathrm{AU}(\mathcal{H}))}_{\text{antiunit. reps. of } \mathbb{R}^\times}, \quad V \mapsto U^V.$$

Application: The Brunetti–Guido–Longo (BGL) construction
If $(U, \mathcal{H})$ is an antiunitary representation of (G, G_+), we obtain a map

$$\mathrm{Hom}_{\mathrm{gr}}(\mathbb{R}^\times, G) \ni \gamma \mapsto V_\gamma \in \mathrm{Stand}(\mathcal{H})$$

determined uniquely by $U^{V_\gamma} = U \circ \gamma \colon \mathbb{R}^\times \to \mathrm{AU}(\mathcal{H})$, i.e.,

$$J_\gamma = U(\gamma(-1)), \quad \Delta_\gamma^{-it/2\pi} = U(\gamma(e^t)), \quad V_\gamma = \mathrm{Fix}(J_\gamma \Delta_\gamma^{1/2}).$$

Nets of real subspaces from distribution vectors

Let $U : G \to \mathrm{AU}(\mathcal{H})$ be an antiunitary representation.
$\mathcal{H}^\infty \subseteq \mathcal{H}$ (smooth vectors), $U^\xi \colon G \to \mathcal{H}, g \mapsto U(g)\xi$ smooth.
$\mathcal{H}^\infty$ carries a natural Fréchet topology, which defines a "rigging"

$$\mathcal{H}^\infty \hookrightarrow \mathcal{H} \xrightarrow{\eta} \mathcal{H}^{-\infty}, \quad \eta(\xi) = |\xi\rangle$$

$\mathcal{H}^{-\infty}$ (distribution vectors) = continuous antilinear functionals on $\mathcal{H}^\infty$.
Any test function $\varphi \in C_c^\infty(G, \mathbb{C})$ defines a smearing operator

$$U^{-\infty}(\varphi) \colon \mathcal{H}^{-\infty} \to \mathcal{H}, \quad \eta \mapsto \int_G \varphi(g) \underbrace{(\eta \circ U(g)^{-1})}_{U^{-\infty}(g)\eta} \, dg \in \mathcal{H}.$$

Let $\mathtt{E} \subseteq \mathcal{H}^{-\infty}$ be a real subspace. Then

$$\mathsf{H}^G_{\mathtt{E}}(\mathcal{O}) := \overline{U^{-\infty}(C_c^\infty(\mathcal{O}, \mathbb{R}))\mathtt{E}} \subseteq \mathcal{H}, \quad \mathcal{O} \subseteq G \ \text{ open}$$

defines a G-covariant isotone net of closed real subspaces of $\mathcal{H}$.

Let $M := G/P$ be a homogeneous space and $q_M \colon G \to G/P, g \mapsto gP$ be the quotient map. Then

$$\mathsf{H}^M_{\mathtt{E}}(\mathcal{O}) := \mathsf{H}^G_{\mathtt{E}}(q_M^{-1}(\mathcal{O})) \subseteq \mathcal{H}, \qquad \mathcal{O} \subseteq M \quad \text{open}$$

is a G-covariant net of closed real subspaces on M.

Problems:

- How do the two constructions of real subspaces match?
- For which data $\mathtt{E}, U, \mathcal{O}$ is the subspace $\mathsf{H}^M_{\mathtt{E}}(\mathcal{O}) \subseteq \mathcal{H}$ standard?
- For a graded homomorphism $\gamma \colon \mathbb{R}^\times \to G$, find domains $\mathcal{O}_\gamma \subseteq M$ with $\mathsf{H}^M_{\mathtt{E}}(\mathcal{O}_\gamma) = V_\gamma$.

Assumptions:

- $C_U := \{x \in \mathfrak{g} \colon -i\partial U(x) \geq 0\}, \quad \partial U(x) = \frac{d}{dt}\big|_{t=0} U(\exp tx)$ (positive cone of U) is pointed (ker U discrete) and generating ("positive energy representations")
- $G = G_+ \rtimes \{\mathbf{1}, \tau\}$, τ involution on G (needed for BGL construction).
- $\gamma \colon \mathbb{R}^\times \to G$, $\gamma(-1) = \tau$, $h = \gamma'(1) \in \mathfrak{g}^\tau$ (graded homo.).

Which pairs (h, τ) are interesting?

For $U \colon G \to \mathrm{AU}(\mathcal{H})$ and $\gamma \colon \mathbb{R}^\times \to G$ as above, define $V := V_\gamma \subseteq \mathcal{H}$ by the BGL construction: $J_V = U(\tau)$, $\Delta_V = e^{2\pi i \cdot \partial U(h)}$.
The order on the homogeneous space $\mathcal{W}(U, \gamma) := U(G_+)V_\gamma \subseteq \mathrm{Stand}(\mathcal{H})$ is encoded in the closed subsemigroup

$$S_V = \{g \in G_+ \colon U(g)V \subseteq V\} \subseteq G_+.$$

Important information on S_V is contained in its Lie wedge

$$\mathbf{L}(S_V) := \{x \in \mathfrak{g} \colon (\forall t \geq 0)\ \exp(tx) \in S_V\},$$

the infinitesimal generators of one-parameter subsemigroups.

Theorem (Structure Theorem for S_V), N. 2020, 2021)

(a) *If $\mathbf{L}(S_V)$ has interior points, then $h = \gamma'(1) \in \mathfrak{g}$ is an Euler element, i.e., $\mathfrak{g} = \mathfrak{g}_1(h) \oplus \mathfrak{g}_0(h) \oplus \mathfrak{g}_{-1}(h)$ for $\mathfrak{g}_j(h) = \{x : [h, x] = jx\}$, and $\mathrm{Ad}(\gamma(-1)) = \mathrm{Ad}(\tau) = e^{\pi i \operatorname{ad} h}$ ($= \mathrm{diag}(-1, 1, -1)$ w.r.t. grading).*

(b) *If h is Euler and $\tau = e^{\pi i \operatorname{ad} h}$, then, for $C_\pm = \pm C_U \cap \mathfrak{g}_{\pm 1}(h)$, we have $S_V = G_V \exp(C_+ + C_-) = \exp(C_+) G_V \exp(C_-)$ (Koufany decomp.).*

The preceding theorem points at the **interesting** homomorphisms $\gamma\colon \mathbb{R}^\times \to G = G_+ \rtimes \{\mathbf{1}, \tau\}$:
$h = \gamma'(1)$ should be an Euler element and $\mathrm{Ad}(\tau) = e^{\pi i \,\mathrm{ad}\, h}$
(see recent paper with V. Morinelli, CMP, 2021).

Euler elements in simple Lie algebras:
$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ Cartan dec., $\mathfrak{a} \subseteq \mathfrak{p}$ maximal abelian with restricted root system

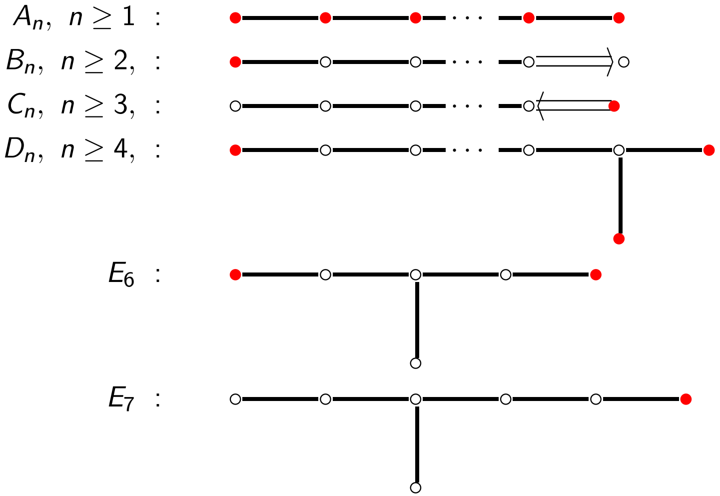
$$\Sigma = \Sigma(\mathfrak{g}, \mathfrak{a}) \supseteq \{\alpha_1, \ldots, \alpha_n\} \quad \text{(simple roots)}$$

and $\alpha_i(h_j) = \delta_{ij}$ (dual basis). Then, for every Euler element h, the adjoint orbit contains a unique h_j. Here is a list of those h_j which are Euler elements (Bourbaki numbering of simple roots):

$$A_n : h_1, \ldots, h_n, \qquad B_n : h_1, \qquad C_n : h_n, \qquad D_n : h_1, h_{n-1}, h_n,$$

$$E_6 : h_1, h_6, \quad E_7 : h_7; \qquad \text{none for} \quad BC_n, E_8, F_4, G_2.$$

Red dots = simple roots α_j for which h_j is Euler, i.e., defines a 3-grading of the root system.



Compactly causal symmetric spaces

Let σ be an involutive automorphism of G, $H = (G_+)^\sigma$ and $M := G_+/H$ the corresponding symmetric space.

Assume: $-\sigma(C_U) = C_U$ and $\sigma(h) = h$, h Euler element.

Causal structure on M: $C_{gH} = g.C, \quad C := C_U \cap \mathfrak{g}^{-\sigma} \subseteq \mathfrak{g}^{-\sigma} \cong T_{eH}(M)$

Tube domain of M: $\mathcal{T}_M := \bigcup_{m \in M} \mathrm{Exp}_m(iC_m^0) = G.\mathrm{Exp}_{eH}(iC^0) \subseteq M_{\mathbb{C}}$

Modular flow on M: $\alpha_t^M(m) = \exp(th)m$, generated by $X_h^M \in \mathcal{V}(M)$, $M^\alpha \subseteq M$ its fixed points.

Three associated "wedge domains":

- $W_M^+(h) := \{m \in M \colon X_h^M(m) \in C_m^0\}$ (positivity domain of the modular flow, X_h^M "timelike").
- $W_M(h) := \bigcup_{m \in M^\alpha} \mathrm{Exp}_m(C_{m,+}^0 + C_{m,-}^0)$, $C_{m,\pm} := \pm C_m \cap T_m(M)_{\pm 1}$ (polar wedge domain).
- $W_M^{\mathrm{KMS}}(h)$: Elements m with $\exp(i(0,\pi)h)m \in \mathcal{T}_M$ (the tube domain). (KMS wedge domain).

Theorem (Ólafsson, N., '21)

For any reductive compactly causal symmetric space $M = G/G^\sigma$ and any Euler element $h \in \mathfrak{g}^\sigma$, all three domains coincide:

$$W_M^+(h) = W_M(h) = W_M^{\mathrm{KMS}}(h).$$

The connected components of these subsets correspond to those of M^α.

Theorem (Existence of nice covariant nets; N./Ólafsson 2020/21)

Let $(U, \mathcal{H})$ be an antiunitary representation of the reductive group G with C_U pointed and generating and $\eta \in (\mathcal{H}^{-\infty})^H$ be an H-invariant cyclic distribution vector fixed by $J := U(\tau)$. Then

$$\mathsf{H}^M_{\mathbb{R}\eta}(\underbrace{W_M(h)_{eH}}_{conn.cpt.}) = V_\gamma \quad \textit{for} \quad \gamma(e^t) = \exp(th), \quad \gamma(-1) = \tau.$$

Ex's: (a) $M = G$, $W_M(h)_e = G_e^h \cdot \exp(C_+^0 + C_-^0)$ (Olshanski semigroup)
(b) $M = \mathrm{AdS}^d \subseteq \mathbb{R}^{2,d-1}$ (anti-de Sitter space),
$G = \mathrm{SO}_{2,d-1}(\mathbb{R})_e$, $H = \mathrm{SO}_{1,d-1}(\mathbb{R})_e$.

Irreducible compactly causal symmetric Lie algebras $(\mathfrak{g}, \tau)$ with Euler element in $\mathfrak{h} = \mathfrak{g}^\tau$:

Group type: $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{h}$, $\mathfrak{h}$ simple hermitian of tube type
$\mathfrak{h} = \mathfrak{su}_{r,r}(\mathbb{C}), \mathfrak{sp}_{2r}(\mathbb{R}), \mathfrak{so}_{2,d}(\mathbb{R}), \mathfrak{so}^*(4r), \mathfrak{e}_{7(-25)}$

Cayley type spaces: $h \in \mathfrak{z}(\mathfrak{h})$ Euler element

$\mathfrak{g}$	$\mathfrak{su}_{r,r}(\mathbb{C})$	$\mathfrak{sp}_{2r}(\mathbb{R})$	$\mathfrak{so}_{2,d}(\mathbb{R}), d > 2$	$\mathfrak{so}^*(4r)$	$\mathfrak{e}_{7(-25)}$
$\mathfrak{h}$	$\mathbb{R} \oplus \mathfrak{sl}_r(\mathbb{C})$	$\mathbb{R} \oplus \mathfrak{sl}_r(\mathbb{R})$	$\mathbb{R} \oplus \mathfrak{so}_{1,d-1}(\mathbb{R})$	$\mathbb{R} \oplus \mathfrak{sl}_r(\mathbb{H})$	$\mathbb{R} \oplus \mathfrak{e}_{6(-26)}$

Split types: $\tau \neq \tau_h$, $\mathrm{rk}_{\mathbb{R}}\mathfrak{h} = \mathrm{rk}_{\mathbb{R}}\mathfrak{g}$.

$\mathfrak{g}$	$\mathfrak{su}_{r,r}(\mathbb{C})$	$\mathfrak{so}_{2,p+q}(\mathbb{R})$	$\mathfrak{so}^*(4r)$	$\mathfrak{e}_{7(-25)}$
$\mathfrak{h}$	$\mathfrak{so}_{r,r}(\mathbb{R})$	$\mathfrak{so}_{1,p}(\mathbb{R}) \oplus \mathfrak{so}_{1,q}(\mathbb{R})$	$\mathfrak{so}_{2r}(\mathbb{C})$	$\mathfrak{sl}_4(\mathbb{H})$

Non-split types: $\tau \neq \tau_h$, $\mathrm{rk}_{\mathbb{R}}\mathfrak{h} = \frac{\mathrm{rk}_{\mathbb{R}}\mathfrak{g}}{2}$.

$\mathfrak{g}$	$\mathfrak{su}_{2s,2s}(\mathbb{C})$	$\mathfrak{sp}_{4s}(\mathbb{R})$	$\mathfrak{so}_{2,d}(\mathbb{R})$
$\mathfrak{h}$	$\mathfrak{u}_{s,s}(\mathbb{H})$	$\mathfrak{sp}_{2s}(\mathbb{C})$	$\mathfrak{so}_{1,d}(\mathbb{R})$

Perspectives 1: Non-compactly causal symmetric spaces

We have presented the part of the theory dealing with spectral conditions $C_U \neq \{0\}$ (related to ground states in Physics). There is a richer, more complicated (thermal) side of the theory, where the order on $U(G_+)V_\gamma \subseteq \mathrm{Stand}(\mathcal{H})$ is trivial and the semigroup $S_{V_\gamma} = G_{V_\gamma}$ is a group.

Ex: $M = \mathrm{dS}^d = \{(x_0, \mathbf{x}) \in \mathbb{R}^{1,d} \colon x_0^2 - \mathbf{x}^2 = -1\}$ (de Sitter space), $W_M^+(h) = \{(x_0, \mathbf{x}) \in M \colon x_1 > |x_0|\}$.

The natural class of spaces to consider here are non-compactly causal symmetric spaces. They carry a nice global order structure, but the wedge domains are more complicated (work in progress with G. Ólafsson).

Exs: $M = G_{\mathbb{C}}/G$, $\mathfrak{g}$ simple hermitian Lie algebra.

Rem.: There is a duality $M \leftrightarrow M^c$ between compactly causal and non-compactly causal symmetric spaces

- $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q} \leftrightarrow \mathfrak{g}^c = \mathfrak{h} \oplus i\mathfrak{q}$ (multiplication of "coordinates by i")
- de Sitter $\leftrightarrow$ anti-de Sitter,
- group type spaces $M = G \leftrightarrow M^c = G_{\mathbb{C}}/G$.

The group case

Assume that $\mathfrak{g}$ is reductive, $(\rho, \mathcal{K})$ is an irreducible antiunitary representation with C_ρ pointed and generating, the Euler element h and $\tau = \tau_h$ are as above and $C_\pm = \pm C_U \cap \mathfrak{g}_{\pm 1}(h)$.

Then we obtain an antiunitary rep. $(U, \mathcal{H}_\rho)$ of $(G \times G) \rtimes \{\mathbf{1}, \tau \times \tau\}$ by

- $\mathcal{H}_\rho := \overline{\rho(C_c^\infty(G, \mathbb{C}))} \subseteq B_2(\mathcal{K})$.
- $U(g_1, g_2)A = \rho(g_1)A\rho(g_2)$, $J(A) = J_\mathcal{K} A J_\mathcal{K}$.
- $W_G((h, h)) = G^h \exp(C_+^0 + C_-^0)$ is a semigroup.
- $\eta(A) := \mathrm{tr}(A)$ is a distribution vector, fixed by Δ_G and J.
- $\mathrm{H}^G_{\mathbb{R}\eta}(W_G((h, h))_e) = V_\gamma$ for $\gamma(e^t) = (\exp th, \exp th), \gamma(-1) = \tau \times \tau$.

$G \cong (G \times G)/\Delta_G$ is a causal symmetric space:
biinvariant cone field $(C_g)_{g \in G}$ determined by $C_e = C_U$.
Conjugation action $\alpha_t(g) = \exp(th) g \exp(-th)$ on the semigroup S implements the modular group of the corresponding von Neumann algebras.

Conformal case

$\mathfrak{g}$ simple hermitian,
$h \in \mathfrak{g}$ Euler element, $\mathrm{Ad}(\tau) = e^{\pi i \,\mathrm{ad}\, h}$
$A = \mathfrak{g}_1(h)$ simple euclidean Jordan algebra,
$M = G/P$ conformal completion of A,
$P := G^h \exp(\mathfrak{g}_{-1}(h))$ (parabolic subgroup)

Theorem

Let $(U, \mathcal{H})$ be an irreducible antiunitary representation of G with C_U pointed and generating. Then there exists a finite dimensional P-invariant subspace $\mathtt{E} \subseteq \mathcal{H}^{-\infty}$ such that

$$\mathrm{H}_{\mathtt{E}}^{G/P}(\exp(C_+^0)P) = V_\gamma \quad \text{for} \quad \gamma(e^t) = \exp(th), \quad \gamma(-1) = \tau.$$

$C_+ \subseteq A$ is the positive cone (of squares) in A.
Affine context: $A \rtimes G^h$ acting on A
Ex.: Poincaré group in the conformal group $G = \mathrm{SO}_{2,d}(\mathbb{R})_e$ of $\mathbb{R}^{1,d-1}$.

Non-reductive Lie groups

- Classification of triples $(\mathfrak{g}, h, C)$, where $h \in \mathfrak{g}$ is an Euler element and $C \subseteq \mathfrak{g}$ an $\mathrm{Ad}(G)$-invariant pointed convex cone for which the cones $C_\pm := \pm C \cap \mathfrak{g}_{\pm 1}(h)$ generate $\mathfrak{g}_{\pm 1}(h)$ (Daniel Oeh, 2020).

  A mixed example: (V, ω) a symplectic vector space, $\mathfrak{heis}(V, \omega) = \mathbb{R} \oplus V$ with $[(z, v), (z', v')] = (\omega(v, v'), 0)$ (Heisenberg algebra)

  $$\mathfrak{g} = \mathfrak{heis}(V, \omega) \rtimes (\mathbb{R}\,\mathrm{id}_V \oplus \mathfrak{sp}(V, \omega)).$$

  Write $V = V_+ \oplus V_-$ for Lagrangian subspaces $V_\pm$.
  Then $\mathfrak{g}$ is 3-graded:

  $$\mathfrak{g}_0 \cong V_- \oplus \mathfrak{gl}(V_-) \oplus \mathbb{R}\,\mathrm{id}_V, \;\; \mathfrak{g}_1 = \mathbb{R} \oplus V_+ \oplus \mathfrak{sp}(V, \Omega)_1, \;\; \mathfrak{g}_{-1} = \mathfrak{sp}(V, \Omega)_{-1}.$$

  $\mathfrak{g}_1 =$ polynomial functions of degree ≤ 2 on V_-.
- Construction of covariant nets on the corresponding groups for general Lie algebras (Daniel Oeh, 2021).

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