

Renormalization in Quantum Field Theory (after R. Borcherds)

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Afternoon Representation Theory Institut Élie Cartan de Lorraine, Metz

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Herscovich Renormalization in QFT (after R. Borcherds)

From:

(i) $k = \mathbb{R}$ or \mathbb{C} ;

Herscovich Renormalization in QFT (after R. Borcherds) Perturbative QFT (after R. Borcherds)

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(ii) *M* is a smooth manifold, provided with a causal order $\leq \subseteq M \times M$, *i.e.* partial order that is closed;

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- (ii) M is a smooth manifold, provided with a causal order $\preceq \subseteq M \times M$, *i.e.* partial order that is closed; \checkmark This allows to define spacelike-separated subsets of M!

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- (i) $k = \mathbb{R}$ or \mathbb{C} ;
- (ii) M is a smooth manifold, provided with a causal order $\leq \subseteq M \times M$, *i.e.* partial order that is closed;
- (iii) $A = C^{\infty}(M,k);$

Perturbative QFT (after R. Borcherds) Herscovich Renormalization in QFT (after R. Borcherds)

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- (ii) *M* is a smooth manifold, provided with a causal order $\leq M \times M$, *i.e.* partial order that is closed;
- (iii) $A = C^{\infty}(M, k)$; \mathcal{N} A comm. *k*-algebra!

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- (ii) *M* is a smooth manifold, provided with a causal order $\leq M \times M$, *i.e.* partial order that is closed;
- (iii) $A = C^{\infty}(M,k)$;
- (iv) *E* is a (super) vector bundle over M;

Herscovich Renormalization in QFT (after R. Borcherds) Perturbative QFT (after R. Borcherds) 2

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- (iv) *E* is a vector bundle over M; so for fixed $i \in \mathbb{N}$ we set $X = \Gamma(J^i E)$; \mathcal{N} A proj. f.g. *A*-module! [Serre-Swan]

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- (v) $V_* = \Gamma_*(\operatorname{Vol}(M))$ and $* \in \{ \ , c\};$

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- (iv) E is a vector bundle over M; so for fixed $i \in \mathbb{N}$ we set $X = \Gamma(J^i E)$;
- (v) $V_* = \Gamma_*(\operatorname{Vol}(M))$, where $\operatorname{Vol}(M) = \Lambda^{\operatorname{top}} T^* M \otimes \mathfrak{o}_M$ and $* \in \{ , c \}$;

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- (v) $V_* = \Gamma_*(Vol(M))$ and $* \in \{ , c \}$; \mathcal{N} A proj. A-module! [Finney-Rotman]

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1. $S_A X$;

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1.
$$S_A X \ (= \oplus_{n=0}^{\infty} X^{\otimes_A n} / \sim);$$

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we consider:

1. $S_A X$; \mathcal{N} "Composite fields or Lagrangians"

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we consider:

- **1**. $S_A X$;
- 2. $\mathscr{L}_* = V_* \otimes_A S_A X;$

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 and $* \in \{ , c \};$

we consider:

1.
$$S_A X$$
;

2. $\mathscr{L}_* = V_* \otimes_A S_A X$; \mathscr{N} "Lagrangians densities"

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 and $* \in \{ , c \};$

we consider:

1. $S_A X$; 2. $\mathscr{L}_* = V_* \otimes_A S_A X$; 3. $S\mathscr{L}_*$;

 Herscovich
 Renormalization in QFT (after R. Borcherds)
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- 3. $S\mathscr{L}_* \ (= \oplus_{n=0}^{\infty} \mathscr{L}_*^{\tilde{\otimes}_{\beta} n} / \sim);$

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 and $* \in \{ , c \};$

we consider:

1. $S_A X$; 2. $\mathscr{L}_* = V_* \otimes_A S_A X$; 3. $S\mathscr{L}_*$; \mathcal{N} "Nonlocal actions (of cpt. supp. if * = c)"

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we consider:

- **1**. $S_A X$;
- 2. $\mathscr{L}_* = V_* \otimes_A S_A X;$
- 3. $S\mathscr{L}_*$; \mathcal{N} The mult. gives the time ordered product!

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3.
$$S\mathscr{L}_*$$
;

4. $T_0(S\mathscr{L}_c);$

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we consider:

1. $S_A X$; 2. $\mathscr{L}_* = V_* \otimes_A S_A X;$ 3. $S\mathscr{L}_*$; 4. $T_0(S\mathscr{L}_c) \ (= \bigoplus_{m=0}^{\infty} (S\mathscr{L}_c)^{\tilde{\otimes}_{\beta} 2m});$

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- 3. $S\mathscr{L}_{*};$
- 4. $T_0(S\mathscr{L}_c)$; \mathcal{N} The mult. gives the composition product!

A propagator Δ is a separately continuous bilinear map

 $\Delta: \Gamma_c\big(\operatorname{Vol}(M)\otimes J^iE\big)\times \Gamma_c\big(\operatorname{Vol}(M)\otimes J^iE\big)\to \mathbb{C},$

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or, equivalently, an element of

 $\operatorname{Hom}_{A\otimes_{\beta}A}\left(X\otimes_{\beta}X,V_{c}^{\prime}\tilde{\otimes}_{\beta}V_{c}^{\prime}\right)$

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Hom_{$$A \otimes_{\beta} A$$} $(X \otimes_{\beta} X, V'_{c} \otimes_{\beta} V'_{c})$
Space of distributions $\mathscr{D}'(M \times M)$

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 $\operatorname{Hom}_{A\otimes_{\beta}A}\left(X\otimes_{\beta}X,V_{c}^{\prime}\tilde{\otimes}_{\beta}V_{c}^{\prime}\right)\simeq \mathscr{D}^{\prime}\left(M\times M,(J^{i}E\boxtimes J^{i}E)^{*}\right).$

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A propagator Δ is *precut* w.r.t. to proper closed convex cones $\mathscr{P}_p \subseteq T_p^*M \ (p \in M)$, if (roughly)

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A propagator Δ is precut w.r.t. to proper closed convex cones $\mathscr{P}_p \subseteq T_p^*M \ (p \in M)$, if (i) if $(v,w) \in WF_{(p,q)}(\Delta)$, for any $(p,q) \in M \times M$, then $-v \in \mathscr{P}_p$ and $w \in \mathscr{P}_q$;

A propagator Δ is a separately continuous bilinear map

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A propagator Δ is precut w.r.t. to proper closed convex cones $\mathscr{P}_p \subseteq T_p^*M \ (p \in M)$, if

- (i) if $(v,w) \in \mathrm{WF}_{(p,q)}(\Delta)$, for any $(p,q) \in M \times M$, then $-v \in \mathscr{P}_p$ and $w \in \mathscr{P}_q$;
- (ii) if $(v,w) \in WF_{(p,p)}(\Delta)$, for any $p \in M$, then w = -v.

Any precut propagator Δ induces a unique map $\tilde{\Delta} \in \operatorname{Hom}_{A \otimes_{\beta} A} \left(S_A X \otimes_{\beta} S_A X, V'_c \otimes_{\beta} V'_c \right)$ such that for all $\sigma_1, \ldots, \sigma_n, \tau_1, \ldots, \tau_m \in X$ and $n, m \in \mathbb{N}$ $\tilde{\Delta}(1_M, 1_M) = 1_{M^2}, \quad \tilde{\Delta}(1_M, \sigma_1 \ldots \sigma_n) = 0,$ $\tilde{\Delta}(\sigma_1 \ldots \sigma_n, \tau_1 \ldots \tau_m) = \delta_{n,m} \sum_{\varsigma \in \mathbb{S}_n} \Delta(\sigma_1, \tau_{\varsigma(1)}) \ldots \Delta(\sigma_n, \tau_{\varsigma(n)}).$

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Borcherds (roughly) claims that $\tilde{\Delta}$ extends to a (unique) Laplace pairing

$$\hat{\Delta} \in \operatorname{Hom}_{\Sigma A \otimes_{\beta} \Sigma A} \left(S(S_A X) \otimes_{\beta} S(S_A X), \Sigma V'_c \tilde{\otimes}_{\beta} \Sigma V'_c \right).$$

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$$\hat{\Delta} \in \operatorname{Hom}_{\Sigma A \otimes_{\beta} \Sigma A} \left(\underbrace{S(S_A X)}_{A \text{ 'bialgebra''?}} \otimes_{\beta} S(S_A X), \underbrace{\Sigma V_c' \tilde{\otimes}_{\beta} \Sigma V_c'}_{An \text{ 'algebra''?}} \right)$$

Any precut propagator Δ induces a unique map

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such that for all $\sigma_1, \ldots, \sigma_n, \tau_1, \ldots, \tau_m \in X$ and $n, m \in \mathbb{N}$

$$\tilde{\Delta}(1_M, 1_M) = 1_{M^2}, \quad \tilde{\Delta}(1_M, \sigma_1 \dots \sigma_n) = 0,$$
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Borcherds (roughly) claims that $\tilde{\Delta}$ extends to a (unique) Laplace pairing

$$\hat{\Delta} \in \operatorname{Hom}_{\Sigma A \otimes_{\beta} \Sigma A} \left(S(S_A X) \otimes_{\beta} S(S_A X), \Sigma V'_c \tilde{\otimes}_{\beta} \Sigma V'_c \right).$$

Borcherds then uses this map in his definition of *Feynman measure* (*FM*) associated with Δ , which is a cont. linear map $\omega : S\mathscr{L}_c \to \mathbb{C}$ satisfying $\omega(1) = 1$, and a recursiveness property involving $\hat{\Delta}$.

Given a symmetric monoidal category $(\mathscr{C}, \otimes, I, \tau)$, a unitary and counitary bialgebra $(C, \mu_C, \Delta_C, 1_C, \varepsilon_C)$ in \mathscr{C} and an unitary algebra $(A, \mu_A, 1_A)$, a *Laplace pairing* is a map $\langle , \rangle : C \otimes C \to A$ in \mathscr{C} such that

$$\langle cc', d \rangle = \langle c, d_{(1)} \rangle \cdot \langle c', d_{(2)} \rangle, \quad \langle c, dd' \rangle = \langle c_{(1)}, d \rangle \cdot \langle c_{(2)}, d' \rangle,$$

 $\langle 1_C, c \rangle = \langle c, 1_C \rangle = \mathcal{E}_C(c) \mathbf{1}_A,$

for all $c, c', d, d' \in C$, where $\Delta_C(c) = c_{(1)} \otimes c_{(2)}$ and $\Delta_C(d) = d_{(1)} \otimes d_{(2)}$ denotes the coproduct of C.

Herscovich Renormalization in QFT (after R. Borcherds) The problem of the Laplace pairing

Given a symmetric monoidal category $(\mathscr{C}, \otimes, I, \tau)$, a unitary and counitary bialgebra $(C, \mu_C, \Delta_C, 1_C, \varepsilon_C)$ in \mathscr{C} and an unitary algebra $(A, \mu_A, 1_A)$, a Laplace pairing is a map $\langle , \rangle : C \otimes C \to A$ in \mathscr{C} such that

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Questions: What is the sym. monoidal category in the claim of Borcherds?

Herscovich Renormalization in QFT (after R. Borcherds) The problem of the Laplace pairing

Given a symmetric monoidal category $(\mathscr{C}, \otimes, I, \tau)$, a unitary and counitary bialgebra $(C, \mu_C, \Delta_C, 1_C, \varepsilon_C)$ in \mathscr{C} and an unitary algebra $(A, \mu_A, 1_A)$, a Laplace pairing is a map $\langle , \rangle : C \otimes C \to A$ in \mathscr{C} such that

$$\langle cc', d \rangle = \langle c, d_{(1)} \rangle \cdot \langle c', d_{(2)} \rangle, \quad \langle c, dd' \rangle = \langle c_{(1)}, d \rangle \cdot \langle c_{(2)}, d' \rangle,$$

 $\langle 1_C, c \rangle = \langle c, 1_C \rangle = \mathcal{E}_C(c) \mathbf{1}_A,$

for all $c, c', d, d' \in C$, where $\Delta_C(c) = c_{(1)} \otimes c_{(2)}$ and $\Delta_C(d) = d_{(1)} \otimes d_{(2)}$ denotes the coproduct of C.

Questions: What is the sym. monoidal category in the claim of Borcherds? **None!**

Herscovich Renormalization in QFT (after R. Borcherds) The problem of the Laplace pairing

Given a symmetric monoidal category $(\mathscr{C}, \otimes, I, \tau)$, a unitary and counitary bialgebra $(C, \mu_C, \Delta_C, 1_C, \varepsilon_C)$ in \mathscr{C} and an unitary algebra $(A, \mu_A, 1_A)$, a Laplace pairing is a map $\langle , \rangle : C \otimes C \to A$ in \mathscr{C} such that

$$\langle cc', d \rangle = \langle c, d_{(1)} \rangle \cdot \langle c', d_{(2)} \rangle, \quad \langle c, dd' \rangle = \langle c_{(1)}, d \rangle \cdot \langle c_{(2)}, d' \rangle,$$

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(1) $S(S_A \mathscr{L}_c)$ has a k-linear product, *i.e.* over \otimes ;

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HerscovichRenormalization in QFT (after R. Borcherds)The problem of the Laplace pairing5 / 16

A *double monoidal category* is a tuple $(\mathscr{C}, \otimes_{\mathscr{C}}, I_{\otimes}, \boxtimes_{\mathscr{C}}, I_{\boxtimes})$, where $(\mathscr{C}, \otimes_{\mathscr{C}}, I_{\otimes})$ and $(\mathscr{C}, \boxtimes_{\mathscr{C}}, I_{\boxtimes})$ are monoidal categories.

Herscovich Renormalization in QFT (after R. Borcherds) The problem of the Laplace pairing

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 $\operatorname{sh}_{A,B,C,D}: (A \otimes_{\mathscr{C}} B) \boxtimes_{\mathscr{C}} (C \otimes_{\mathscr{C}} D) \to (A \boxtimes_{\mathscr{C}} C) \otimes_{\mathscr{C}} (B \boxtimes_{\mathscr{C}} D)$

in ${\mathscr C}$ and three morphisms

$$\begin{split} \mu_\boxtimes: I_\otimes\boxtimes_{\mathscr C} I_\otimes \to I_\otimes, \quad \Delta_\otimes: I_\boxtimes \to I_\boxtimes \otimes_{\mathscr C} I_\boxtimes \quad \text{and} \quad \nu: I_\boxtimes \to I_\otimes, \\ \text{in } \mathscr C \text{ satisfying several "natural" compatibility conditions.} \end{split}$$

Herscovich Renormalization in QFT (after R. Borcherds) The problem of the Laplace pairing

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 $\mu_{\boxtimes}: I_{\otimes}\boxtimes_{\mathscr{C}} I_{\otimes} \to I_{\otimes}, \quad \Delta_{\otimes}: I_{\boxtimes} \to I_{\boxtimes} \otimes_{\mathscr{C}} I_{\boxtimes} \quad \text{ and } \quad v: I_{\boxtimes} \to I_{\otimes},$

in \mathscr{C} satisfying several "natural" compatibility conditions. \mathcal{N} It means that it is a pseudomonoid in $\ell(Cat)$.

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Example 2.

Any symmetric monoidal category $(\mathscr{C}, \otimes, I, \tau)$ is 2-monoidal with $\otimes_{\mathscr{C}} = \otimes = \boxtimes_{\mathscr{C}}, I_{\otimes} = I = I_{\boxtimes}$ and $\mathrm{sh} = \mathrm{id} \otimes \tau \otimes \mathrm{id}$.

Herscovich Renormalization in QFT (after R. Borcherds) The problem of the Laplace pairing

Let $(\mathscr{C}, \otimes_{\mathscr{C}}, I_{\otimes}, \boxtimes_{\mathscr{C}}, I_{\boxtimes}, \mathrm{sh})$ be a 2-monoidal category.

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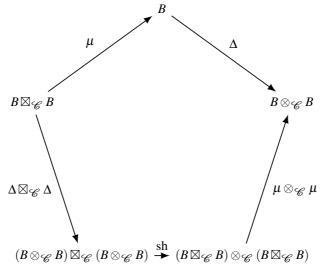
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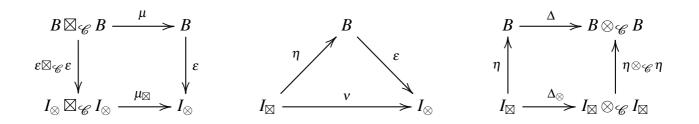
such that





Bialgebras in 2-monoidal categories (cont.)

and

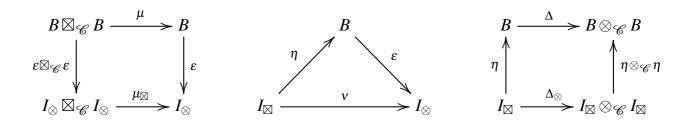


commute.

Herscovich Renormalization in QFT (after R. Borcherds) The problem of the Laplace pairing

Bialgebras in 2-monoidal categories (cont.)

and



commute.

Still an issue: there is no definition of Laplace pairing for bialgebras in 2-monoidal categories!

Herscovich	Renormalization in QFT (after R. Borcherds)	The problem of the Laplace pairing	8 / 16
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A symmetric 2-monoidal category $(\mathscr{C}, \otimes_{\mathscr{C}}, I_{\otimes}, \tau, \boxtimes_{\mathscr{C}}, I_{\boxtimes}, sh)$ is called *framed* [H., '17] if there are: (a) a symmetric monoidal category $(\mathscr{C}', \boxtimes_{\mathscr{C}'}, I'_{\boxtimes}, \tau')$;

Herscovich Renormalization in QFT (after R. Borcherds) The problem of the Laplace pairing

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- (a) a symmetric monoidal category $(\mathscr{C}', \boxtimes_{\mathscr{C}'}, I'_{\boxtimes}, \tau')$;
- (b) a faithful functor $F : \mathscr{C} \to \mathscr{C}'$ that is symmetric lax monoidal w.r.t. $(\mathscr{C}, \otimes_{\mathscr{C}}, I_{\otimes}, \tau)$ (and coherence maps φ_0 and φ_2), and strict
 - monoidal w.r.t. $(\mathscr{C}, oxtimes_{\mathscr{C}}, I_oxtimes)$ (and coherence maps ψ_0 and ψ_2);

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Herscovich Renormalization in QFT (after R. Borcherds) The problem of the Laplace pairing

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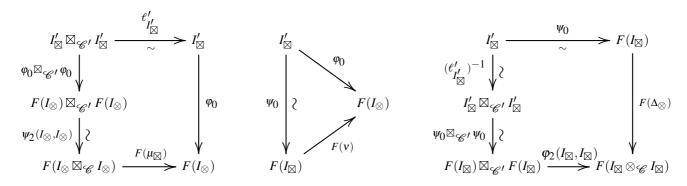
- (a) a symmetric monoidal category $(\mathscr{C}', \boxtimes_{\mathscr{C}'}, I'_{\boxtimes}, \tau')$;
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- w.r.t. $(\mathscr{C}, \otimes_{\mathscr{C}}, I_{\otimes}, \tau)$ (and coherence maps φ_0 and φ_2), and strict monoidal w.r.t. $(\mathscr{C}, \boxtimes_{\mathscr{C}}, I_{\boxtimes})$ (and coherence maps ψ_0 and ψ_2); such that

commutes for all objects A, B, C and D in \mathscr{C} ,

Herscovich Renormalization in QFT (after R. Borcherds) The problem of the Laplace pairing 9 / 16

Framed 2-monoidal categories (cont.)

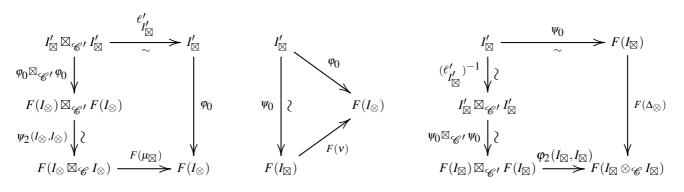
as well as



Herscovich Renormalization in QFT (after R. Borcherds) The problem of the Laplace pairing

Framed 2-monoidal categories (cont.)

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Theorem 3 (H., '17).

Let A be a unit. comm. alg. in a cocomplete sym. mon. cat. $(\mathscr{C}, \otimes_{\mathscr{C}}, I_{\mathscr{C}}, \tau)$ such that $\otimes_{\mathscr{C}}$ commutes with colimits on each side. Let $B = {}^{\mu}TA$ be the comm. counit. bialg. in \mathscr{C} with deconcatenation coproduct and the tensor-wise product of A. Then, the category ${}_{B}\operatorname{Mod}(\mathscr{C})$ of (firm) modules over B in \mathscr{C} has natural structure of framed 2-monoidal category, where \otimes is given by \otimes_{B} and \boxtimes by $\otimes_{\mathscr{C}}$.

Herscovich Renormalization in QFT (after R. Borcherds) The problem of the Laplace pairing

Consider

(i) $(\mathscr{C}, \otimes_{\mathscr{C}}, I_{\otimes}, \tau, \boxtimes_{\mathscr{C}}, I_{\boxtimes}, \operatorname{sh})$ a sym. 2-monoidal category framed inside of $(\mathscr{C}', \boxtimes_{\mathscr{C}'}, I'_{\boxtimes}, \tau')$ via the functor F;

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- (ii) a unit. and counit. bialgebra $(C, \mu_C, \eta_C, \Delta_C, \varepsilon_C)$ relative to \mathscr{C} ;

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- (iii) a unit. algebra $(A, \mu_{A,\ell}, \eta_{A,\ell})$ in $(\mathscr{C}, \otimes_{\mathscr{C}}, I_{\otimes})$.

Herscovich Renormalization in QFT (after R. Borcherds) The problem of the Laplace pairing

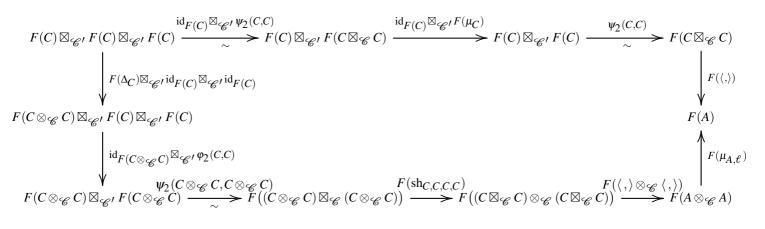
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Herscovich Renormalization in QFT (after R. Borcherds) The problem of the Laplace pairing

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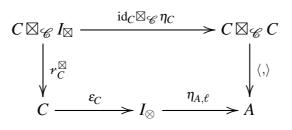
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- (ii) a unit. and counit. bialgebra $(C, \mu_C, \eta_C, \Delta_C, \varepsilon_C)$ relative to \mathscr{C} ; (iii) a unit. algebra $(A, \mu_{A,\ell}, \eta_{A,\ell})$ in $(\mathscr{C}, \otimes_{\mathscr{C}}, I_{\otimes})$.
- A *left Laplace pairing* [H., '17] on C relative to \mathscr{C} and with values on A is a map $\langle , \rangle : C \boxtimes_{\mathscr{C}} C \to A$ in \mathscr{C} such that



commutes in \mathscr{C}'

HerscovichRenormalization in QFT (after R. Borcherds)The problem of the Laplace pairing11 / 16

and



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Solution to both problems

Theorem 4 (H., '17).

The construction $T(S_AX)$ has a natural structure of bialgebra relative to the framed sym. 2-monoidal category μ_{TA} Mod, whose product is given by concatenation and whose coproduct is induced by that of S_AX (using the interchange law). Moreover, $\tilde{\Delta}$ extends to a left (resp., right) Laplace pairing

 $\hat{\Delta} \in \operatorname{Hom}_{{}^{\mu}TA \otimes_{\beta}{}^{\mu}TA} \left(T(S_{A}X) \otimes_{\beta} T(S_{A}X), TV_{c}' \tilde{\otimes}_{\beta} TV_{c}' \right).$

Herscovich Renormalization in QFT (after R. Borcherds) The problem of the Laplace pairing

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It has two different algebra structures! ightarrow

Herscovich Renormalization in QFT (after R. Borcherds) The problem of the Laplace pairing

Given a local precut propagator Δ , let \mathscr{F}_{Δ} be the set of all FM associated with Δ .

Herscovich Renormalization in QFT (after R. Borcherds) The construction of the QFT

Given a local precut propagator Δ , let \mathscr{F}_{Δ} be the set of all FM associated with Δ .

Define the *renormalization group* \mathscr{G} as the subgroup of the group of automorphisms of the cocomm. coalgebra $S_A \mathscr{L}$ in $_A$ Mod that are $S_A X$ -colinear.

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Theorem 5 (Borcherds, '11; H., '17).

The group \mathscr{G} has a natural action on \mathscr{F}_{Δ} that is simple and transitive.

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Theorem 6 (Borcherds, '11; H., '17).

Under further assumptions on Δ , namely Δ is of cut type and manageable, \mathscr{F}_{Δ} is nonempty.

Herscovich Renormalization in QFT (after R. Borcherds) The construction of the QFT

Fact 7 (Borcherds, '11; H.).

Given any Feynman measure $\omega : S\mathscr{L}_c \to \mathbb{C}$, there exists a unique extension to a continuous linear map $\breve{\omega} : T(S\mathscr{L}_c) \to \mathbb{C}$ satisfying a certain recursiveness property.

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The restriction $\mathring{\omega}$ of $\breve{\omega}$ to $T_0(S\mathscr{L}_c)$ is called a *free QFT*.

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The restriction $\mathring{\omega}$ of $\breve{\omega}$ to $T_0(S\mathscr{L}_c)$ is called a free QFT.

Theorem 8 (Borcherds, '11; H.).

The restriction $\mathring{\omega}$ of the canonical extension $\breve{\omega}: T(S\mathscr{L}_c) \to \mathbb{C}$ of a FM associated to a local manageable propagator of cut type is equal on commutators of elements of $T_0(S\mathscr{L}_c)$ whose supports are spacelike-separated.

Herscovich Renormalization in QFT (after R. Borcherds) The construction of the QFT

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The bialgebra $S\mathscr{L}_c$ acts naturally on the algebra $T_0(S\mathscr{L}_c)$.

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An *interacting QFT* is the continuous and $\mathbb{C}[[\lambda]]$ -linear map $\Omega_I : T_0(S\mathscr{L}_c \otimes \mathbb{C}[[\lambda]]) \to \mathbb{C}[[\lambda]]$ given as the composition of $\mathring{\omega}$ and $\exp(iL_I)$.

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