Renormalization in Quantum Field Theory (after R. Borcherds)

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Afternoon Representation Theory
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December 17th, 2020

Basic data

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(ii) M is a smooth manifold, provided with a causal order $\preceq \subseteq M \times M$, *i.e.* partial order that is closed; $\rightsquigarrow$ This allows to define spacelike-separated subsets of M!

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1. $S_A X$; $\rightsquigarrow$ "Composite fields or Lagrangians"

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1. $S_A X$;
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3. $S\mathscr{L}_*$; $\rightsquigarrow$ The mult. gives the time ordered product!

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4. $T_0(S\mathscr{L}_c)$; $\rightsquigarrow$ The mult. gives the composition product!

Basic definitions I: Propagators

A *propagator* Δ is a separately continuous bilinear map

$$\Delta : \Gamma_c\big(\mathrm{Vol}(M) \otimes J^i E\big) \times \Gamma_c\big(\mathrm{Vol}(M) \otimes J^i E\big) \to \mathbb{C},$$

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Space of distributions $\mathscr{D}'(M \times M)$

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(ii) if $(v, w) \in \mathrm{WF}_{(p,p)}(\Delta)$, for any $p \in M$, then $w = -v$.

Fact 1.

Any precut propagator Δ *induces a unique map*

$$\tilde{\Delta} \in \mathrm{Hom}_{A \otimes_\beta A}\left(S_A X \otimes_\beta S_A X, V_c' \tilde{\otimes}_\beta V_c'\right)$$

such that for all $\sigma_1, \dots, \sigma_n, \tau_1, \dots, \tau_m \in X$ *and* $n, m \in \mathbb{N}$

$$\tilde{\Delta}(1_M, 1_M) = 1_{M^2}, \quad \tilde{\Delta}(1_M, \sigma_1 \dots \sigma_n) = 0,$$
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Borcherds (roughly) claims that $\tilde{\Delta}$ extends to a (unique) Laplace pairing

$$\hat{\Delta} \in \mathrm{Hom}_{\Sigma A \otimes_\beta \Sigma A}\left(S(S_A X) \otimes_\beta S(S_A X), \Sigma V_c' \tilde{\otimes}_\beta \Sigma V_c'\right).$$

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Borcherds then uses this map in his definition of *Feynman measure (FM) associated with Δ*, which is a cont. linear map $\omega : S\mathscr{L}_c \to \mathbb{C}$ satisfying $\omega(1) = 1$, and a recursiveness property involving $\hat{\Delta}$.

The (first) definition of a Laplace pairing

Given a symmetric monoidal category $(\mathscr{C}, \otimes, I, \tau)$, a unitary and counitary bialgebra $(C, \mu_C, \Delta_C, 1_C, \varepsilon_C)$ in $\mathscr{C}$ and an unitary algebra $(A, \mu_A, 1_A)$, a *Laplace pairing* is a map $\langle \, , \rangle : C \otimes C \to A$ in $\mathscr{C}$ such that

$$\langle cc', d \rangle = \langle c, d_{(1)} \rangle \cdot \langle c', d_{(2)} \rangle, \quad \langle c, dd' \rangle = \langle c_{(1)}, d \rangle \cdot \langle c_{(2)}, d' \rangle,$$
$$\langle 1_C, c \rangle = \langle c, 1_C \rangle = \varepsilon_C(c) 1_A,$$

for all $c, c', d, d' \in C$, where $\Delta_C(c) = c_{(1)} \otimes c_{(2)}$ and $\Delta_C(d) = d_{(1)} \otimes d_{(2)}$ denotes the coproduct of C.

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Questions: What is the sym. monoidal category in the claim of Borcherds?

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Reasons:

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Questions: What is the sym. monoidal category in the claim of Borcherds? $\rightsquigarrow$ None!

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(1) $S(S_A \mathscr{L}_c)$ has a k-linear product, *i.e.* over $\otimes$;

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$$\langle 1_C, c\rangle = \langle c, 1_C\rangle = \varepsilon_C(c) 1_A,$$

for all $c, c', d, d' \in C$, where $\Delta_C(c) = c_{(1)} \otimes c_{(2)}$ and $\Delta_C(d) = d_{(1)} \otimes d_{(2)}$ denotes the coproduct of C.

Questions: What is the sym. monoidal category in the claim of Borcherds? $\rightsquigarrow$ None!

Reasons:

(1) $S(S_A \mathscr{L}_c)$ has a k-linear product, *i.e.* over $\otimes$;

(2) $S(S_A \mathscr{L}_c)$ should have *a priori* a coproduct with respect to $\otimes_{\Sigma A}$ (or something similar).

The (first) definition of a Laplace pairing

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(2) $S(S_A \mathscr{L}_c)$ should have *a priori* a coproduct with respect to $\otimes_{\Sigma A}$ (or something similar). $\rightsquigarrow$ It doesn't have!

The solution: 2-monoidal categories

A *double monoidal category* is a tuple $(\mathscr{C}, \otimes_{\mathscr{C}}, I_{\otimes}, \boxtimes_{\mathscr{C}}, I_{\boxtimes})$, where $(\mathscr{C}, \otimes_{\mathscr{C}}, I_{\otimes})$ and $(\mathscr{C}, \boxtimes_{\mathscr{C}}, I_{\boxtimes})$ are monoidal categories.

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A *2-monoidal category* [Aguiar-Mahajan, '10; Batanin-Markl, '12; Street, '12] is a double monoidal category provided with

$$\mathrm{sh}_{A,B,C,D} : (A \otimes_{\mathscr{C}} B) \boxtimes_{\mathscr{C}} (C \otimes_{\mathscr{C}} D) \to (A \boxtimes_{\mathscr{C}} C) \otimes_{\mathscr{C}} (B \boxtimes_{\mathscr{C}} D)$$

in $\mathscr{C}$ and three morphisms

$$\mu_{\boxtimes} : I_{\otimes} \boxtimes_{\mathscr{C}} I_{\otimes} \to I_{\otimes}, \quad \Delta_{\otimes} : I_{\boxtimes} \to I_{\boxtimes} \otimes_{\mathscr{C}} I_{\boxtimes} \quad \text{and} \quad \nu : I_{\boxtimes} \to I_{\otimes},$$

in $\mathscr{C}$ satisfying several "natural" compatibility conditions.

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in $\mathscr{C}$ satisfying several "natural" compatibility conditions. $\rightsquigarrow$ It means that it is a pseudomonoid in $\ell(\mathrm{Cat})$.

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Example 2.

Any symmetric monoidal category $(\mathscr{C}, \otimes, I, \tau)$ is 2-monoidal with $\otimes_{\mathscr{C}} = \otimes = \boxtimes_{\mathscr{C}}$, $I_{\otimes} = I = I_{\boxtimes}$ and $\mathrm{sh} = \mathrm{id} \otimes \tau \otimes \mathrm{id}$.

Bialgebras in 2-monoidal categories

Let $(\mathscr{C}, \otimes_{\mathscr{C}}, I_{\otimes}, \boxtimes_{\mathscr{C}}, I_{\boxtimes}, \mathrm{sh})$ be a 2-monoidal category.

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(1) a unitary alg. struct. (B, μ, η) w.r.t. $(\mathscr{C}, \boxtimes_{\mathscr{C}}, I_{\boxtimes})$;

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(2) a counitary coalg. struct. (B, Δ, ε) w.r.t. $(\mathscr{C}, \otimes_{\mathscr{C}}, I_{\otimes})$;
such that

$$\begin{tikzcd}
 & B \arrow[dr, "\Delta"] & \\
B \boxtimes_{\mathscr{C}} B \arrow[ur, "\mu"] \arrow[d, "\Delta \boxtimes_{\mathscr{C}} \Delta"'] & & B \otimes_{\mathscr{C}} B \\
(B \otimes_{\mathscr{C}} B) \boxtimes_{\mathscr{C}} (B \otimes_{\mathscr{C}} B) \arrow[rr, "\mathrm{sh}"] & & (B \boxtimes_{\mathscr{C}} B) \otimes_{\mathscr{C}} (B \boxtimes_{\mathscr{C}} B) \arrow[u, "\mu \otimes_{\mathscr{C}} \mu"']
\end{tikzcd}$$

Bialgebras in 2-monoidal categories (cont.)

and

$$
\begin{tikzcd}
B \boxtimes_{\mathscr{C}} B \arrow[r, "\mu"] \arrow[d, "\varepsilon \boxtimes_{\mathscr{C}} \varepsilon"'] & B \arrow[d, "\varepsilon"] \\
I_{\otimes} \boxtimes_{\mathscr{C}} I_{\otimes} \arrow[r, "\mu_{\boxtimes}"] & I_{\otimes}
\end{tikzcd}
\qquad
\begin{tikzcd}
 & B \arrow[dr, "\varepsilon"] & \\
I_{\boxtimes} \arrow[ur, "\eta"] \arrow[rr, "\nu"] & & I_{\otimes}
\end{tikzcd}
\qquad
\begin{tikzcd}
B \arrow[r, "\Delta"] & B \otimes_{\mathscr{C}} B \\
I_{\boxtimes} \arrow[u, "\eta"] \arrow[r, "\Delta_{\otimes}"] & I_{\boxtimes} \otimes_{\mathscr{C}} I_{\boxtimes} \arrow[u, "\eta \otimes_{\mathscr{C}} \eta"']
\end{tikzcd}
$$

commute.

Bialgebras in 2-monoidal categories (cont.)

and

$$
\begin{tikzcd}
B \boxtimes_{\mathscr{C}} B \arrow[r, "\mu"] \arrow[d, "\varepsilon \boxtimes_{\mathscr{C}} \varepsilon"'] & B \arrow[d, "\varepsilon"] \\
I_{\otimes} \boxtimes_{\mathscr{C}} I_{\otimes} \arrow[r, "\mu_{\boxtimes}"] & I_{\otimes}
\end{tikzcd}
\qquad
\begin{tikzcd}
 & B \arrow[dr, "\varepsilon"] & \\
I_{\boxtimes} \arrow[ur, "\eta"] \arrow[rr, "\nu"] & & I_{\otimes}
\end{tikzcd}
\qquad
\begin{tikzcd}
B \arrow[r, "\Delta"] & B \otimes_{\mathscr{C}} B \\
I_{\boxtimes} \arrow[u, "\eta"] \arrow[r, "\Delta_{\otimes}"] & I_{\boxtimes} \otimes_{\mathscr{C}} I_{\boxtimes} \arrow[u, "\eta \otimes_{\mathscr{C}} \eta"']
\end{tikzcd}
$$

commute.

Still an issue: there is no definition of Laplace pairing for bialgebras in 2-monoidal categories!

Solution: Framed 2-monoidal categories

A symmetric 2-monoidal category $(\mathscr{C}, \otimes_{\mathscr{C}}, I_{\otimes}, \tau, \boxtimes_{\mathscr{C}}, I_{\boxtimes}, \mathrm{sh})$ is called *framed* [H., '17] if there are:

(a) a symmetric monoidal category $(\mathscr{C}', \boxtimes_{\mathscr{C}'}, I'_{\boxtimes}, \tau')$;

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(b) a faithful functor $F : \mathscr{C} \to \mathscr{C}'$ that is symmetric lax monoidal w.r.t. $(\mathscr{C}, \otimes_{\mathscr{C}}, I_{\otimes}, \tau)$ (and coherence maps φ_0 and φ_2), and strict monoidal w.r.t. $(\mathscr{C}, \boxtimes_{\mathscr{C}}, I_{\boxtimes})$ (and coherence maps ψ_0 and ψ_2);

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such that

$$\begin{tikzcd}
F(A) \boxtimes_{\mathscr{C}'} F(B) \boxtimes_{\mathscr{C}'} F(C) \boxtimes_{\mathscr{C}'} F(D) \arrow[r, "\varphi_2(A,B) \boxtimes_{\mathscr{C}'} \varphi_2(C,D)"] \arrow[d, "\mathrm{id}_{F(A)} \boxtimes_{\mathscr{C}'} \tau'_{\boxtimes}(F(B),F(C)) \boxtimes_{\mathscr{C}'} \mathrm{id}_{F(D)}"'] & F(A \otimes_{\mathscr{C}} B) \boxtimes_{\mathscr{C}'} F(C \otimes_{\mathscr{C}} D) \arrow[d, "\psi_2(A \otimes_{\mathscr{C}} B, C \otimes_{\mathscr{C}} D)"', "\wr"] \\
F(A) \boxtimes_{\mathscr{C}'} F(C) \boxtimes_{\mathscr{C}'} F(B) \boxtimes_{\mathscr{C}'} F(D) \arrow[d, "\psi_2(A,C) \boxtimes_{\mathscr{C}'} \psi_2(B,D)", "\wr"'] & F\big((A \otimes_{\mathscr{C}} B) \boxtimes_{\mathscr{C}} (C \otimes_{\mathscr{C}} D)\big) \arrow[d, "F(\mathrm{sh}_{A,B,C,D})"] \\
F(A \boxtimes_{\mathscr{C}} C) \boxtimes_{\mathscr{C}'} F(B \boxtimes_{\mathscr{C}} D) \arrow[r, "\varphi_2(A \boxtimes_{\mathscr{C}} C, D \boxtimes_{\mathscr{C}} D)"] & F\big((A \boxtimes_{\mathscr{C}} C) \otimes_{\mathscr{C}} (B \boxtimes_{\mathscr{C}} D)\big)
\end{tikzcd}$$

commutes for all objects A, B, C and D in $\mathscr{C}$,

Framed 2-monoidal categories (cont.)

as well as

$$
\begin{tikzcd}
I'_{\boxtimes} \boxtimes_{\mathscr{C}'} I'_{\boxtimes} \arrow[rr, "\ell'_{I'_{\boxtimes}}", "\sim"'] \arrow[d, "\varphi_0 \boxtimes_{\mathscr{C}'} \varphi_0"'] & & I'_{\boxtimes} \arrow[dd, "\varphi_0"] \\
F(I_{\otimes}) \boxtimes_{\mathscr{C}'} F(I_{\otimes}) \arrow[d, "\psi_2(I_{\otimes}, I_{\otimes})"', "\wr"] & & \\
F(I_{\otimes} \boxtimes_{\mathscr{C}} I_{\otimes}) \arrow[rr, "F(\mu_{\boxtimes})"] & & F(I_{\otimes})
\end{tikzcd}
\qquad
\begin{tikzcd}
I'_{\boxtimes} \arrow[dr, "\varphi_0"] \arrow[dd, "\psi_0"', "\wr"] & \\
& F(I_{\otimes}) \\
F(I_{\boxtimes}) \arrow[ur, "F(\nu)"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
I'_{\boxtimes} \arrow[r, "\psi_0", "\sim"'] \arrow[d, "(\ell'_{I'_{\boxtimes}})^{-1}"', "\wr"] & F(I_{\boxtimes}) \arrow[dd, "F(\Delta_{\otimes})"] \\
I'_{\boxtimes} \boxtimes_{\mathscr{C}'} I'_{\boxtimes} \arrow[d, "\psi_0 \boxtimes_{\mathscr{C}'} \psi_0"', "\wr"] & \\
F(I_{\boxtimes}) \boxtimes_{\mathscr{C}'} F(I_{\boxtimes}) \arrow[r, "\varphi_2(I_{\boxtimes}, I_{\boxtimes})"] & F(I_{\boxtimes} \otimes_{\mathscr{C}} I_{\boxtimes})
\end{tikzcd}
$$

Framed 2-monoidal categories (cont.)

as well as

$$\begin{tikzcd}
I'_{\boxtimes} \boxtimes_{\mathscr{C}'} I'_{\boxtimes} \arrow[rr, "\ell'_{I'_{\boxtimes}}", "\sim"'] \arrow[d, "\varphi_0 \boxtimes_{\mathscr{C}'} \varphi_0"'] & & I'_{\boxtimes} \arrow[dd, "\varphi_0"] \\
F(I_{\otimes}) \boxtimes_{\mathscr{C}'} F(I_{\otimes}) \arrow[d, "\psi_2(I_{\otimes}, I_{\otimes})"', "\wr"] & & \\
F(I_{\otimes} \boxtimes_{\mathscr{C}} I_{\otimes}) \arrow[rr, "F(\mu_{\boxtimes})"] & & F(I_{\otimes})
\end{tikzcd}
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\end{tikzcd}
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I'_{\boxtimes} \boxtimes_{\mathscr{C}'} I'_{\boxtimes} \arrow[d, "\psi_0 \boxtimes_{\mathscr{C}'} \psi_0"', "\wr"] & & \\
F(I_{\boxtimes}) \boxtimes_{\mathscr{C}'} F(I_{\boxtimes}) \arrow[rr, "\varphi_2(I_{\boxtimes}, I_{\boxtimes})"] & & F(I_{\boxtimes} \otimes_{\mathscr{C}} I_{\boxtimes})
\end{tikzcd}$$

Theorem 3 (H., '17).

Let A be a unit. comm. alg. in a cocomplete sym. mon. cat. $(\mathscr{C}, \otimes_{\mathscr{C}}, I_{\mathscr{C}}, \tau)$ such that $\otimes_{\mathscr{C}}$ commutes with colimits on each side. Let $B = {}^{\mu}TA$ be the comm. counit. bialg. in $\mathscr{C}$ with deconcatenation coproduct and the tensor-wise product of A. Then, the category ${}_B\mathrm{Mod}(\mathscr{C})$ of (firm) modules over B in $\mathscr{C}$ has natural structure of framed 2-monoidal category, where $\otimes$ is given by $\otimes_B$ and $\boxtimes$ by $\otimes_{\mathscr{C}}$.

Laplace pairings in framed 2-monoidal categories

Consider

(i) $(\mathscr{C}, \otimes_{\mathscr{C}}, I_{\otimes}, \tau, \boxtimes_{\mathscr{C}}, I_{\boxtimes}, \mathrm{sh})$ a sym. 2-monoidal category framed inside of $(\mathscr{C}', \boxtimes_{\mathscr{C}'}, I'_{\boxtimes}, \tau')$ via the functor F;

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(ii) a unit. and counit. bialgebra $(C, \mu_C, \eta_C, \Delta_C, \varepsilon_C)$ relative to $\mathscr{C}$;

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(ii) a unit. and counit. bialgebra $(C, \mu_C, \eta_C, \Delta_C, \varepsilon_C)$ relative to $\mathscr{C}$;

(iii) a unit. algebra $(A, \mu_{A,\ell}, \eta_{A,\ell})$ in $(\mathscr{C}, \otimes_{\mathscr{C}}, I_{\otimes})$.

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(iii) a unit. algebra $(A, \mu_{A,\ell}, \eta_{A,\ell})$ in $(\mathscr{C}, \otimes_{\mathscr{C}}, I_{\otimes})$.

A *left Laplace pairing* [H., '17] on C relative to $\mathscr{C}$ and with values on A is a map $\langle\,,\rangle : C \boxtimes_{\mathscr{C}} C \to A$ in $\mathscr{C}$ such that

$$\begin{tikzcd}
F(C)\boxtimes_{\mathscr{C}'}F(C)\boxtimes_{\mathscr{C}'}F(C) \arrow[r, "\mathrm{id}_{F(C)}\boxtimes_{\mathscr{C}'}\psi_2(C,C)", "\sim"'] \arrow[d, "F(\Delta_C)\boxtimes_{\mathscr{C}'}\mathrm{id}_{F(C)}\boxtimes_{\mathscr{C}'}\mathrm{id}_{F(C)}"] & F(C)\boxtimes_{\mathscr{C}'}F(C\boxtimes_{\mathscr{C}}C) \arrow[r, "\mathrm{id}_{F(C)}\boxtimes_{\mathscr{C}'}F(\mu_C)"] & F(C)\boxtimes_{\mathscr{C}'}F(C) \arrow[r, "\psi_2(C,C)", "\sim"'] & F(C\boxtimes_{\mathscr{C}}C) \arrow[d, "F(\langle,\rangle)"] \\
F(C\otimes_{\mathscr{C}}C)\boxtimes_{\mathscr{C}'}F(C)\boxtimes_{\mathscr{C}'}F(C) \arrow[d, "\mathrm{id}_{F(C\otimes_{\mathscr{C}}C)}\boxtimes_{\mathscr{C}'}\varphi_2(C,C)"] & & & F(A) \\
F(C\otimes_{\mathscr{C}}C)\boxtimes_{\mathscr{C}'}F(C\otimes_{\mathscr{C}}C) \arrow[r, "\psi_2(C\otimes_{\mathscr{C}}C,C\otimes_{\mathscr{C}}C)", "\sim"'] & F\big((C\otimes_{\mathscr{C}}C)\boxtimes_{\mathscr{C}}(C\otimes_{\mathscr{C}}C)\big) \arrow[r, "F(\mathrm{sh}_{C,C,C,C})"] & F\big((C\boxtimes_{\mathscr{C}}C)\otimes_{\mathscr{C}}(C\boxtimes_{\mathscr{C}}C)\big) \arrow[r, "F(\langle\,,\rangle\otimes_{\mathscr{C}}\langle\,,\rangle)"] & F(A\otimes_{\mathscr{C}}A) \arrow[u, "F(\mu_{A,\ell})"']
\end{tikzcd}$$

commutes in $\mathscr{C}'$

Laplace pairings in framed 2-monoidal categories (cont.)

and

$$\begin{tikzcd}
C \boxtimes_{\mathscr{C}} I_{\boxtimes} \arrow[rr, "\mathrm{id}_C \boxtimes_{\mathscr{C}} \eta_C"] \arrow[d, "r^{\boxtimes}_C"] & & C \boxtimes_{\mathscr{C}} C \arrow[d, "\langle , \rangle"] \\
C \arrow[r, "\varepsilon_C"] & I_{\otimes} \arrow[r, "\eta_{A,\ell}"] & A
\end{tikzcd}$$

commutes in $\mathscr{C}$.

Solution to both problems

Theorem 4 (H., '17).

The construction $T(S_A X)$ has a natural structure of bialgebra relative to the framed sym. 2-monoidal category ${}_{\mu_{TA}}\mathrm{Mod}$, whose product is given by concatenation and whose coproduct is induced by that of $S_A X$ (using the interchange law). Moreover, $\tilde{\Delta}$ extends to a left (resp., right) Laplace pairing

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It has two different algebra structures!

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Under further assumptions on Δ, namely Δ is of cut type and manageable, $\mathscr{F}_\Delta$ is nonempty.

The construction of the QFT (cont.)

Fact 7 (Borcherds, '11; H.).

Given any Feynman measure $\omega : S\mathscr{L}_c \to \mathbb{C}$, there exists a unique extension to a continuous linear map $\breve{\omega} : T(S\mathscr{L}_c) \to \mathbb{C}$ satisfying a certain recursiveness property.

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Theorem 8 (Borcherds, '11; H.).

The restriction $\mathring{\omega}$ of the canonical extension $\breve{\omega} : T(S\mathscr{L}_c) \to \mathbb{C}$ of a FM associated to a local manageable propagator of cut type is equal on commutators of elements of $T_0(S\mathscr{L}_c)$ whose supports are spacelike-separated.

The construction of the QFT (cont.)

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An *interacting QFT* is the continuous and $\mathbb{C}[[\lambda]]$-linear map $\Omega_I : T_0(S\mathscr{L}_c \otimes \mathbb{C}[[\lambda]]) \to \mathbb{C}[[\lambda]]$ given as the composition of $\mathring{\omega}$ and $\exp(iL_I)$.