Resonances
on Symmetric Spaces

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Resolvents

(M^d, g) .... complete Riemannian manifold

$\Delta = \delta d = \nabla^* \nabla$ ... positive Laplace operator on functions on M

$R_z = (\Delta - z)^{-1}$ ... resolvent regarded as an analytic family of operators $L^2_c(M) \to H^2_{loc}(M)$.

The resolvent may extend meromorphically from the resolvent set across the spectrum.

Resolvents

Question

What is the maximal Riemann surface to which $(\Delta - z)^{-1}$ continues and what are the poles?

Simple examples

$M = \mathbb{R}^{2n+1}$, $n \geq 1$:

Quadratic branched cover of $\mathbb{C}$, i.e. $(\Delta - \lambda^2)^{-1}$ is entire.

Adding a compactly supported metric or topological perturbation gives a meromorphic function with finite rank Laurent coefficients.

Simple examples

$M = \mathbb{R}^{2n}$:

Logarithmic branched cover of $\mathbb{C}$, i.e. $(\Delta - \exp(\lambda))^{-1}$ is entire.

Adding a compactly supported metric or topological perturbation gives a meromorphic function with finite rank Laurent coefficients (even at the logarithmic branching point (J. Müller and A.S. 2014)).

Rank one symmetric spaces

$M = G/K$ a rank one symmetric space

Quadratic branched cover of $\mathbb{C}$, i.e. $(\Delta - \lambda^2 - \rho^2)^{-1}$ is meromorphic with finite rank negative Laurent coefficients. (Guillope-Zworski 1995 (for $\mathbb{H}^n$), Bunke-Olbrich 2000, Carron-Pedon 2002)

Poles are irreducible finite dimensional representations (Guillope-Zworski 1995 ($\mathbb{H}^n$), Zworski 2006 ($\mathbb{H}^n$), Hilgert-Pasquale 2008).

Adding a compactly supported metric or topological perturbation gives a meromorphic function with finite rank Laurent coefficients.

Results for Symmetric Spaces

$M = G/K$ **a higher rank symmetric space, rank odd**

Quadratic branched but incomplete cover of $\mathbb{C}$, i.e.
$(\Delta - \lambda^2 - \rho^2)^{-1}$ is entire for $|\arg \lambda| < \pi/2$.
(Mazzeo-Vasy 2005, A.S. 2005)

Adding a compactly supported metric or topological perturbation gives a meromorphic function with finite rank Laurent coefficients.

In special cases the resolvent continues to a full cover of the complex plane (see Huygen's principle on the symmetric space).

Results for Symmetric Spaces

$M = G/K$ **a higher rank symmetric space, rank even**

Quadratic branched but incomplete cover of $\mathbb{C}$, i.e.
$(\Delta - \exp(\lambda) - \rho^2)^{-1}$ is entire for $|\Im\lambda| < \pi$.
(Mazzeo-Vasy 2005, A.S. 2005)

Adding a compactly supported metric or topological perturbation gives a meromorphic function with finite rank Laurent coefficients.

In the case of $M = SL_3(\mathbb{R})/SO(3)$: further continuation beyond $|\Im\lambda| < \pi$ leads to poles with infinite rank Laurent coefficients (Hilgert, Pasquale, Przebinda 2014)

Infinite multiplicities

$M = M_1 \times M_2$, **where** M_1, M_2 **are irreducible symmetric spaces of non-compact type**

Possibly poles with infinite rank Laurent coefficients if the group G_2 of $M_2 = G_2/K_2$ has infinite dimensional irreducible representations in $L^2(M_2)$.

Two commuting operators!!

A collection of results

M **manifold with a non-compact edge**

Meromorphic continuation of R_z to a neighbourhood of the spectrum. (Cano 2012, using complex scaling)

M **asymptotically hyperbolic**

Quadratic branched cover of $\mathbb{C}$, i.e. $(\Delta - \lambda^2 - \rho^2)^{-1}$ is meromorphic except at a point. (W. Müller 1987 (cusps of $\mathbb{Q}$-rank one), Mazzeo-Melrose 1987, Guillarmou 2005, Vasy 2012).

A collection of results

M **has (asymptotic) cylindrical end of the form** $Y \times \mathbb{R}_+$

Meromorphic on a quadratic branched cover of $\mathbb{C}$ with infinitely many branching points at the eigenvalues of Δ_Y. Poles have finite rank negative Laurent coefficients. (Guillope 1989, Melrose 1993)

A collection of results

M **has a generalized cusp end of the form** $Y \times \mathbb{R}_+$ **with metric of the form** $dx^2 + x^{-2a} g_Y$

Meromorphic on a cover of $\mathbb{C}$ with one branch point that can be either logarithmic or of order p, depending on rationality of a. (Golenia-Moroianu 2008, Hunsicker, Roidos, A.S. 2012). Poles have finite rank Laurent coefficients.

Consequences

Theorem

If $(\Delta - z)^{-1}$ *is meromorphic on a cover of* $\mathbb{C}$ *containing the spectrum, then the spectrum is pure point (at the poles)* $+$ *absolutely continuous.*

A much stronger statement is actually true: on the dense set of the Hilbert space the spectral measure is analytic almost everywhere.

Relation to the spectral measure

The spectral family E_λ and the resolvent family R_z contain the same information:

$$dE_\lambda = \lim_{\epsilon \to 0^+} \frac{1}{2\pi i}(R_{\lambda+i\epsilon} - R_{\lambda-i\epsilon})$$

and

$$R_z = \int_{\mathbb{R}} \frac{1}{\lambda - z} dE_\lambda.$$

The spectral measure on symmetric spaces of non-compact type

For $f, g \in C_0^\infty(M)$ we have

$$\begin{aligned} \langle f, (\Delta_g - z)^{-1} g \rangle = \int_{a^* \times B} \frac{\overline{\hat{f}(\lambda, b)} \hat{g}(\lambda, b)}{|\lambda|^2 + |\rho|^2 - z} \frac{d\lambda}{|c(\lambda)|^2} db \\ \hat{f}(\lambda, b) = \int_M f(x) e^{(-i\lambda + \rho)(A(x,b))} dx, \\ A(gK, kZ) = A(k^{-1} g) \end{aligned} \tag{1}$$

$G = NAK$ Iwasawa decomposition, $g = N(g) exp(A(g)) K(g)$
$B = K/Z$, Z centralizer of A in K.

Huygens' principle

(X, η) ... Lorentzian globally hyperbolic space-time

$\Box$ ... normally hyperbolic differential operator

G_+ retarded fundamental solution

$\mathrm{supp}\, G_+ \subset \{(x, y) \in X \times X \mid$
$x, y \text{ can be connected by a causal curve}\}$

$\Box$ is called **Huygens operator** iff
$\mathrm{supp}\, G_+ \subset \{(x, y) \in X \times X \mid$
$x, y \text{ can be connected by a lightlike geodesic}\}$

local!!!
Old problem by Hadamard: find all Huygens operators!

Huygens' principle and the resolvent

(M, g) ... complete simply connected non-positively curved Riemannian manifold

$\Box = -\frac{\partial^2}{\partial t^2} - P,$

$P = \Delta + V(x)$, $V \in C^\infty(M)$.

Theorem

If $\Box$ is a Huygens operator then $(P - \lambda^2)^{-1}$ is meromorphic with at most a simple pole at $\lambda = 0$.

Huygens' principle and the resolvent, The proof

Proof: For $f, g \in C_0^\infty(M)$ we know that

$$\langle f, \cos(t\sqrt{P})g\rangle$$

is compactly supported in t and therefore

$$\langle f, P^{-1/2}\sin(t\sqrt{P})g\rangle$$

is a constant plus a compactly supported function. The result follows from

$$\langle f, (P-\lambda^2)^{-1}g\rangle = \int_0^\infty \langle f, P^{-1/2}\sin(t\sqrt{P})g\rangle e^{-\lambda t}dt.$$

Relation to scattering theory

Stationary scattering theory can be applied directly when the resolvent is meromorphic:

- Scattering matrix is holomorphic near the spectrum and meromorphic with possible poles away from it.
- Generalized eigenfunctions are meromorphic.
- Analytic continuation of Eisenstein series.
- for congruence surfaces $\Gamma(N)\backslash\mathbb{H}$ the poles of the resolvent are directly related to the non-trivial zeros of Riemann's zeta function.

(only?) Techniques

- Complex scaling
- meromorphic Fredholm theory and parametrix constructions (for example the 0-calculus)
- Pseudolaplacians (Colin de Verdiere)
- Helgason's Fourier transform (spherical Fourier transform), i.e. explicit spectral measure.

Laplace operator on *p*-forms

Similar results available, but branching points depend on p.
Resonances at zero may have cohomological meaning.

- Atiyah-Patodi-Singer 1975,
- for symmetric spaces in various generalities: , Mazzeo-Melrose 1987, Epstein-Mazzeo-Melrose 1991, Bunke-Olbrich 2000, Carron-Pedon 2004
- Carron 2002

Dirac type operators

Similar results as for the Laplacian can be obtained. (Melrose 1992, Guillarmou-Moroianu-Park 2010 for $\mathbb{H}^n$, Carron (2002,2005), W. Müller (1994), Vaillant 2001)

Anosov vector fields, Pollicott-Ruelle resonances

(Ruelle (1985), Pollicott (1986), Baladi-Keller (1990), Gouzel-Liverani (2006), Baladi-Tsujii(2007) , Sjöstrand-Faure (2008), Dyatlov-Zworski (2014), Dyatlov-Faure-Guillarmou (2014))
for symmetric spaces of rank one this links to resonances of the Laplace operator.
locally symmetric spaces of higher rank!!

Spectral measure for tuples of commuting operators

The spectral measure for a a commuting set of operators lives on $\mathbb{R}^n$ and then could be continued analytically to $\mathbb{C}^n$. Is there a notion of resonance in this case?
(higher rank situation)