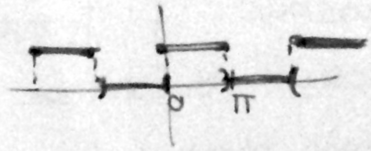
EX $f(x) := \begin{cases} 1 & \text{si } t \in [0, \pi] \\ 0 & \text{si } t \in]\pi, 2\pi[\end{cases}$



2π-périodique

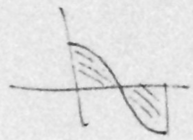
(1) Montrer que f est C^1 par morceaux
(2) Calculer les coeffs de Fourier réels de f
(3) Calculer les coeff de Fourier complexes de f
(4) Calculer la série de Fourier de f
(en forme réelle et en forme complexe)
(5) Calculer la somme de la série de Fourier de f

Corrigé

(1) Ok

(2) Fonction ni paire, ni impaire

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{\pi}\int_0^{\pi} dx = \frac{1}{\pi}[x]_0^{\pi} = 1$$



$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)dx = \frac{1}{\pi}\int_0^{\pi}\cos(nx)dx$$
$(n = 1, 2, 3, \dots)$
$$= \frac{1}{\pi}\left[\frac{\sin(nx)}{n}\right]_0^{\pi} = 0$$

$$b_0 = 0$$
$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)dx = \frac{1}{\pi}\int_0^{\pi}\sin(nx)dx = \frac{1}{\pi}\left[\frac{-\cos(nx)}{n}\right]_0^{\pi}$$
$$= \frac{1}{\pi}\,\frac{-(-1)^n - 1}{n}$$
$$= \frac{1}{\pi}\,\frac{(-1)^{n+1} - 1}{n} = \begin{cases} 0 & n \text{ pair} \\ \frac{-2}{\pi n} & n \text{ impair} = 2k+1 \end{cases}$$

(3) $$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\overbrace{e^{-inx}}^{\cos(nx) - i\sin(nx)}dx$$
$$= \frac{a_n - ib_n}{2} \quad (n \geq 0)$$

$$c_{-n} = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{inx}dx = \frac{a_n + ib_n}{2} \quad (n \geq 0)$$

$$c_0 = \frac{a_0}{2} = \frac{1}{2}$$

$$c_n = \frac{-ib_n}{2} = \begin{cases} 0 & n \text{ pair} \\ \frac{-i}{\pi n} & n = 2k+1 \text{ impair} \end{cases} \quad (n \geq 0)$$

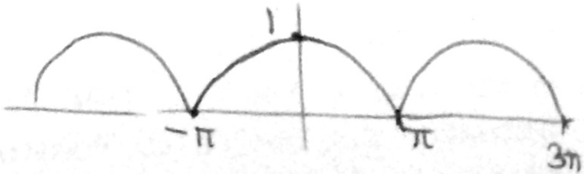
$$c_{-n} = \overline{c_n} = \begin{cases} 0 & n \text{ pair} \\ \frac{+i}{\pi n} & n = 2k+1 \text{ imp.} \end{cases}$$

$$c_n = \begin{cases} 0 & n \text{ pair} \\ \frac{-i}{\pi n} & n \text{ imp.} \end{cases}$$

(4) $$\frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\cos(nx) + b_n\sin(nx))$$
$$= \frac{1}{2} + \sum_{\substack{n=1 \\ n \text{ impair}}}^{\infty}\left(\frac{-2}{\pi n}\right)\sin(nx) = \frac{1}{2} + \sum_{k=0}^{\infty}\frac{-2}{\pi(2k+1)}\sin((2k+1)x)$$
$$= \frac{1}{2} - \frac{2}{\pi}\sum_{k=0}^{\infty}\frac{\sin((2k+1)x)}{2k+1}$$

$$\sum_{n=-\infty}^{+\infty} c_n e^{inx} = \frac{1}{2} + \sum_{k=-\infty}^{+\infty}\frac{-i}{\pi(2k+1)}e^{i(2k+1)x}$$

EX - f 2π-périodique $f : [-\pi, \pi[\to \mathbb{R}$ déf par $f(x) = 1 - \frac{x^2}{\pi^2}$



(1) f C^1 par morceaux et continue

(2) Coeffs de Fourier réels de f

f paire :

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$$
$$= \frac{2}{\pi}\int_0^{\pi}\left(1 - \frac{x^2}{\pi^2}\right)dx$$
$$= \frac{2}{\pi}\left[\pi - \frac{1}{3}\frac{\pi^3}{\pi^2}\right] = \frac{2}{\pi}\left[\frac{2}{3}\pi\right] = \frac{4}{3}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx$$
$$= \frac{2}{\pi}\int_0^{\pi}\left(1 - \frac{x^2}{\pi^2}\right)\cos(nx)\,dx$$
$$= \frac{2}{\pi}\left[\underbrace{\int_0^{\pi}\cos(nx)\,dx}_{\underbrace{\frac{1}{n}\sin(nx)\Big|_0^{\pi}}_{=0}} - \frac{1}{\pi^2}\int_0^{\pi} x^2\cos(nx)\,dx\right]$$
$$= -\frac{2}{\pi^3}\cdot 2\pi(-1)^n$$
$$= (-1)^{n+1}\cdot\frac{4}{\pi^2 n^2}$$

$$\int_0^{\pi} x^2 \overbrace{\cos(nx)}^{\left(\frac{\sin(nx)}{n}\right)'}\,dx = \left[\frac{x^2\sin(nx)}{n}\right]_0^{\pi}$$
$$- \int_0 \frac{2x}{n}\underbrace{\sin(nx)}_{\left[-\frac{\cos(nx)}{n}\right]'}\,dx$$
$$= \left[\frac{2x\cos(nx)}{n^2}\right]_0^{\pi} - 2\int_0^{\pi}\frac{\cos(nx)}{n^2}\,dx$$
$$= \frac{2\pi\cos(n\pi)}{n^2} - 2\left[\frac{\sin(nx)}{n^3}\right]_0^{\pi}$$
$$= 2\pi\frac{(-1)^n}{n^2}$$

$b_n = 0$

(3) Série de Fourier de f

$$\frac{2}{3} + \sum_{n=1}^{\infty}\frac{(-1)^{n+1}4}{\pi^2 n^2}\cos(nx)$$

(4) convergence normale vers $f(x)$

$[x = \pi]$ $$\frac{2}{3} + \sum_{n=1}^{\infty}(-1)^{n+1}\frac{4}{\pi^2 n^2}\overbrace{\cos(n\pi)}^{=(-1)^n} = 1 - 1 = 0$$

$$\frac{2}{3} - \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2} = 0 \implies \sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}$$

$[n = 0]$ $$\frac{2}{3} + \sum_{n=1}^{\infty}\frac{(-1)^{n+1}4}{\pi^2 n^2} = 1$$

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2} = \frac{-\pi^2}{12}$$