Final Exam Review

Chapter 1: Introduction

Section 1.1: Terminology: (ordinary) differerential equation, independent and dependent variables, solutions, integral curves, general solution. Use DE's for modeling: Newton's law of cooling, population growth models (see also Section 2.5). See Section 2.3 for mixing problems.

Section 1.2 and Section 2.5: Autonomous differential equations: find equilibrium solutions (also called critical points or stationary points), draw phase lines, sketch integral curves. Determine if a critical point is asymptotically stable, semistable or unstable. Drawing of direction fields is not requested.

Section 1.3: classification of DE's: order, linear/non linear, homogeneity. Initial value problems (IVP).

Chapter 2: First order differential equations

Section 2.1: solve separable equations.

Section 2.2: standard form of a first order linear DE. Solve first order linear differential equations by using integrating factors.

Section 2.3: modeling: write down a differential equation to model a problem and then solve the differential equation (or the IVP). Mixing problems.

Section 2.4: existence and uniqueness of solutions: first order linear DE (Theorem 2.4.1) and first order non-linear DE (Theorem 2.4.2).

Section 2.5: see Section 1.2.

Section 2.6: recognize a first order exact DE and solve it.

Section 2.7 is not on the final.

Chapter 3: Systems of two first order equations

Section 3.1: System of two linear equations in matrix form. Trace and determinant of a 2×2 matrix. A matrix is invertible if and only if its determinant is non-zero. Inverse of a matrix. Solutions of linear systems. Eigenvalues and eigenvectors.

Section 3.2: IVP for a system of two first-order linear DE's. Matrix notation. Component plots of solutions. Homogenous systems.
Autonomous systems: notions of phase plane, trajectories, direction fields, equilibrium point (or equilibrium point or critical point), phase portrait. See also Section 7.1.
Transform a second order linear DE into a system of first order linear DE's.

Section 3.3: Reduce the non-homogeneous system $\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{b}$ to the homogeous s system $\mathbf{x}' = \mathbf{A}\mathbf{x}$.
For a homogeneous system $\mathbf{x}' = \mathbf{A}\mathbf{x}$: Superposition principle (Theorem 3.3.1), Wronskian and linear independence, fundamental system of solutions, general solution (Theorem 3.3.4).
Sections 3.3, 3.4 and 3.5: Solve the homogeneous system $\mathbf{x}' = \mathbf{A}\mathbf{x}$:
- Find a fundamental system of solutions and write the general solution (depending of the nature of the eigenvalues of $\mathbf{A}$).
- When $\mathbf{A}$ has complex eigenvalues, write the solution in terms of real solutions (Section 3.4).
- Behavior of the solutions when $\mathbf{A}$ has two distinct real eigenvalues (Table 3.3.1) or complex eigenvalues (Table 3.4.1) or repeteated eigenvalues (Table 3.5.1).
- Determine if $(0, 0)$ is a nodal sink, nodal source, saddle, spiral sink, spiral source, or a center. Stability.

Sections 3.6: nonlinear systems of two differential equations. Vector notation. Existence and uniqueness of the solutions (Theorem 3.6.1). Autonomous systems (see also Chapter 7).

Chapter 4: Second order linear equations

Section 4.1: Second order linear equations: standard form, homogeneous/nonhomogenous, constant coefficients/variable coefficients, initial value problems (IVP). See below for spring-mass system models.
Section 4.2: The system of first order linear differential equations associated with a second order linear differential equation, correspondence of initial conditions, matrix notation.

Existence and uniqueness of the solutions of an IVP for a 2nd order linear DE (Theorem 4.2.1).
Second order linear homogenous DE's: principle of superposition for a 2nd order DE (Theorem 4.2.2, Corollary 4.2.3) and for a homogenous system of 1st order linear DE's (Theorem 4.2.4, Corollary 4.2.5). Wronskian of two solutions, fundamental solutions and general solution: for homogenous systems of two 1st order linear DEs (Theorem 4.2.6) and for 2nd order linear DE (Theorem 4.2.7).

Section 4.3: Second order linear homogenous DE's with constant coefficients: characteristic equation, fundamental system of solutions constructed from the roots of the characteristic equation (Theorem 4.3.1, for the DE and for its associated system), general solution (Theorem 4.3.2).

Section 4.5: Solutions of a second order linear nonhomogenous DE: the general solution as a sum of the general solution of the corresponding homogenous DE (complementary solution) and one particular solution (Theorems 4.5.1 and 4.5.2). The method of undetermined coefficients for finding a particular solution when the corresponding homogenous differential equation has constant coefficients.

Sections 4.7: The method of variation of parameters is not on the program of the final. You are welcome to use it instead of the method of undetermined coefficients unless otherwise requested.

Sections 4.1, 4.4 and 4.6: Spring-mass systems
- *Section 4.1:* the model: mass, spring constant, damping factor.
- *Section 4.4:* unforced or free systems (harmonic oscillators).

Undamped free system: phase-amplitude form of the general solution (period, natural frequency, phase, amplitude).
Damped free system: underdamped, critically damped or overdamped harmonic motion; critical damping; quasi-frequency and quasi-period of an underdamped harmonic motion. The phase portraits for harmonic oscillators (pp. 249–250) are not in the final.
- *Section 4.6:* forced systems with periodic external force. The method of complex-valued exponentials $f(t) = Ae^{i\omega t}$. The special case of forced systems without damping.

The notions of transient solutions, steady-state solutions, frequency response function, gain factor, phase shift, resonance are not in the final.

Chapter 5: The Laplace transform

Section 5.1: Improper integrals: examples and tests of convergence (Theorem 5.1.4). Piecewise continuous functions. Functions of exponential order. The Laplace transform $\mathcal{L}$: definition, linearity (Theorem 5.1.2), Laplace transform of piecewise continuous functions of exponential order (Theorem 5.1.6, Corollary 5.1.7).

Section 5.2: Laplace transforms of $e^{ct}f$, of $t^n f(t)$, of derivatives, of differential equations.

Section 5.3: Inverse Laplace transform $\mathcal{L}^{-1}$, linearity, partial fraction decompositions.

Sections 5.4 and 5.6: Solving differential equations with Laplace transforms.

Section 5.5: unit step functions, indicator functions and their Laplace transforms. Representations of piecewise defined functions. Laplace transforms of time-shifted functions. The Laplace transforms of periodic functions are not on the final exam.

Section 5.7: Definition of Dirac's delta distribution δ. Laplace transform of $\delta(t)$ and of $\delta(t - t_0)$.

Section 5.8: Convolution $f * g$ of two piecewise continuous functions f and g on $[0, +\infty)$ (Definition 5.8.1) and the convolution theorem (Theorem 5.8.3).

Chapter 7: Non-linear differential equations and stability

Section 7.1: autonomous systems of two first order DE's. Critical points and stability (precise definitions). The oscillating pendulum.

Section 7.2: Isolated critical points, almost linear system, Jacobian matrix, linear approximation of an almost linear system. Stability and instability properties of almost linear systems in relation to those of their linear approximations (Table 7.2.2). The damped pendulum.

Sections 7.3 and 7.4: Examples of almost linear systems: competing species models and prey-predator equations.

Chapters 6 and 8 are not on the final exam.