Final Review Problems

Exercise 1 Consider the differential equation

$$\frac{dy}{dt} = f(y) \quad \text{with} \quad f(y) = (y^2 - 3)(y + 4)$$

1. Sketch the graph of $f(y)$ versus y.
2. Determine the critical points (or equilibrium solutions) of the differential equation.
3. Classify each critical point as asymptotically stable, unstable or semistable. Draw the phase line and sketch several graphs of solutions in the ty-plane.

Exercise 2

1. State if the following differential equations are linear or nonlinear and solve the given initial value problems:
   (1) $y' = 2y^2 + 2ty^2$ with initial condition $y(0) = 1$.
   (2) $y' = 2y - e^{3t} + t$ with initial condition $y(0) = 1$.
2. For each of the two IVP, determine the largest interval containing $t = 0$ on which the solution is definied.

Exercise 3 Consider the differential equations

$$\begin{array}{ll} \text{a)} & 3x^2y^2\,y' = -(1 + 2xy^3) \\ \text{b)} & (1 + 2xy^3)y' = -3x^2y^2\,. \end{array}$$

1. One of the two differential equations is exact. Which one?
2. Solve the differential equation that is exact.

Exercise 4 Consider the system of linear differential equations $\mathbf{x}' = \mathbf{A}\mathbf{x}$, where $\mathbf{A}$ is one of the following two matrices:

$$(a)\ \begin{pmatrix} -4 & -2 \\ 3 & -11 \end{pmatrix}; \qquad (b)\ \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}$$

For each of them:

1. Find the general solution of the linear system.
2. Determine the equilibrium solutions, their type and stability.
3. Solve the IVP with the inital condition $\mathbf{x}(0) = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$.

Exercise 5 A mass of 2 kilograms is attached to a spring with spring constant 26 N/m. The damping constant γ for the system is 12 N.sec/m. Suppose the mass is moved 0.5 m upward of equilibrium and given an initial upward velocity of 3 m/sec.

1. Determine the initial value problem describing the movement of the mass.
2. Find the position of the mass at any time t.
3. Find the amplitude, the quasi-frequency and the quasi-period of the solution.

Exercise 6 If the Wronskian of f and g is $t\cos t - \sin t$, and if $u = f + 3g, v = f - g$, find the Wronskian of u and v.

Exercise 7 Consider the differential equation $5y'' - 4y' = 0$.

1. Write a system of first order differential equations which is equivalent to the given equation.
2. Determine the general solution of the system in **1**.
3. Determine the general solution of the second order differential equation.
4. What can you say about the critical points?
5. Determine a suitable form for the particular solution $Y(t)$ of the differential equation

$$5y'' - 4y' = t^2 + e^{-4/5t}$$

if the method of undetermined coefficients is to be used. You do not need to find the value of the coefficients or solve the differential equation.

Exercise 8 **1.** Find the Laplace transform of

$$f(t) = -2e^{3t}\cos(t) - \ln(t^2+1)\delta(t-3) + \int_0^t \tau^2 \sin(3(t-\tau))d\tau$$

2. Find the inverse Laplace transform of

$$F(s) = \frac{2}{(s-3)(s-2)(s-1)} - 3\frac{e^{-5s}}{(s+6)^8}.$$

Exercise 9 Consider the piecewise defined function

$$g(t) = \begin{cases} 1 & \text{if } 0 \le t < 2 \\ t-2 & \text{if } 2 \le t \end{cases}$$

1. Express $g(t)$ in terms of unit step functions.
2. Find the solution of the initial value problem

$$y'' + y = g(t) \quad \text{with} \quad y(0) = 0 \text{ and } y'(0) = 0$$

Exercise 10 Consider the following system of differential equations:

$$\begin{aligned} \frac{dx}{dt} &= y(3-x-y) \\ \frac{dy}{dt} &= x(2-x) \end{aligned}$$

1. Find all the critical points of the system.
2. Compute the Jacobian matrix for the system.
3. For each critical point, find the corresponding approximating linear system. Find the eigenvalues of each linear system and classify each critical point according to type (nodal, spiral, center,...) and stability (asymptotically stable, stable, or unstable).