Georgia Tech Lorraine - Metz
MATH 2552
Instructor: *Dr S. Mehdi*

SPRING 2019
Differential Equations
Teaching Assistant: *Mr S. Karrakchou*

FINAL EXAM

Instructions for Students

- Students are required to show their work and justify their answer.
  Unsupported answer will not get any points.
- Organize your work in a reasonably neat and coherent way.
- Calculators, notes, cell phones, books are not allowed.
  Only tables provided by your instructor will be allowed.
- Please ensure you use dark pencil or pen and use clear handwritting.
  Your exam may be scanned into a digital system.
- Pages are single-sided. One side is left intentionally blank for you to show more work if necessary.
- Leave a 1cm border around the edges of exams.

Georgia Tech Honor Code

Having read the Georgia Institute of Technology Academic Honor Code, I understand and accept my responsability as a member of the Georgia Tech community to uphold the Honor Code at all times. In addition, I understand my options for reporting honor violations as detailed in the code.

________________________________ (Name and Signature) __________ (Date)

1. A body of a man is found at 2am with temperature 30^0 Celcius. An hour later the body has temperature 25^0 Celcius. The room temperature is 20^0 Celcius. We assume that the man had fever before dying, his body temperature was 40^0 Celcius.

a. (3 points) Write a differential equation that models the temperature of the body.

b. (5 points) Solve your differential equation to determine an expression for the temperature of the body as a function of time. Make sure to compute the (transmission) coefficient k.
(*You may leave your answer in terms of* ln.)

c. (4 points) When did the person die? (*You may leave your answer in terms of* ln.)

a. Write $y(t)$ for the temperature of the body at time t
(time t will be in hours)
Let $T = 20°C$ be the ambiant temperature.
Newton's law of cooling asserts that:

3 $$\boxed{\frac{dy}{dt} = -k(y-T) \qquad y(0) = 30°C}$$

b. The differential equation is separable

$$\frac{dy}{y-T} = -k\,dt \iff \ln|y-T| = -kt + \text{constant}$$
$$\iff y(t) = T + \lambda e^{-kt} \qquad \lambda \text{ constant}$$

5 $$y(0) = 30 \iff T + \lambda = 30 \iff 20 + \lambda = 30 \iff \lambda = 10$$
$$y(1) = 25 \iff 20 + 10e^{-k} = 25 \iff e^{-k} = \frac{5}{10} = \frac{1}{2} \iff k = \ln(2)$$
$$\boxed{y(t) = 20 + 10e^{-\ln(2)t}}$$

c. We need to find t such that $y(t) = 40$

4 $$y(t) = 40 \iff 20 + 10e^{-\ln(2)t} = 40 \iff e^{-\ln(2)t} = 2 \iff t = -1$$

$\boxed{\text{The man died at 1am}}$.

2. A mass of $\frac{1}{8}$ Kilograms is attached to a spring with spring constant 16 N/m. The damping constant (friction coefficient) for the system is 2 N.sec/m. Suppose the mass is moved 3/4 m upward of equilibrium and given an initial upward velocity of 2 m/sec.

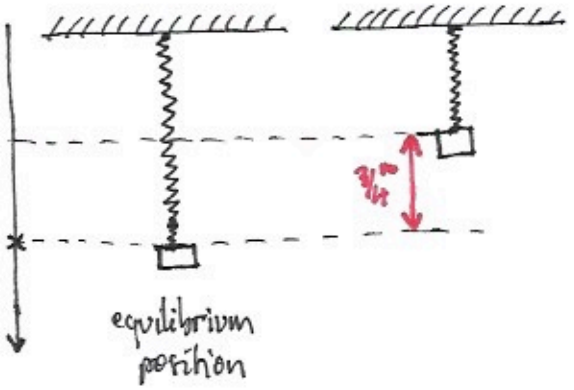
a. (3 points) Determine the initial value problem describing the movement of the mass.

b. (4 points) Find the position of the mass at any time t.

c. (3 points) Find the amplitude, the quasifrequency and the quasiperiod of the solution.

3

a.



No external force

$$\frac{1}{8}y'' + 2y' + 16 = 0$$

$$y(0) = -3/4 \qquad y'(0) = -2$$

4

b. characteristic equation is $\frac{1}{8}r^2 + 2r + 16 = 0$

$\Delta = 4 - 4 \times \frac{1}{8} \times 16 = -4 < 0$ solutions are $\frac{-2 \pm 2i}{\frac{1}{4}} = -8 \pm 8i$

$$y(t) = c_1 e^{-8t}\cos(8t) + c_2 e^{-8t}\sin(8t)$$

$$y(0) = c_1 \Leftrightarrow \boxed{c_1 = -\frac{3}{4}}$$

$$y'(t) = -8c_1 e^{-8t}\cos(8t) - 8c_1 e^{-8t}\sin(8t) - 8c_2 e^{-8t}\sin(8t) + 8c_2 e^{-8t}\cos(8t)$$

$$y'(0) = -8c_1 + 8c_2 \Leftrightarrow -8c_1 + 8c_2 = -2 \Leftrightarrow \boxed{c_2 = -1}$$

$$\boxed{y(t) = -\frac{3}{4}e^{-8t}\cos(8t) - e^{-8t}\sin(8t)}$$

3

c. amplitude is $\sqrt{\frac{9}{16} + 1} = \sqrt{\frac{25}{16}} = \boxed{\frac{5}{4}}$

quasifrequency is $\boxed{8}$

quasi period is $\frac{2\pi}{8} = \boxed{\frac{\pi}{4}}$

3. Clearly show your reasoning.

a. (6 points) Find the Laplace transform of

$$f(t) = e^t\cos(2t) + t\delta(t-2) + \int_0^t (t-\tau)\sin(3\tau)d\tau$$

b. (6 points) Find the inverse Laplace transform of

$$F(s) = \frac{1}{(s+1)(s-2)} + \frac{e^{-s}}{(s-3)^4}$$

(Hint: find real numbers a and b such that $\frac{1}{(s+1)(s-2)} = \frac{a}{s+1} + \frac{b}{s-2}$.)

a. $\mathcal{L}\{e^t \sin(2t)\}(s) = \frac{2}{(s-1)^2 + 2^2}$

$\mathcal{L}\{t\delta(t-2)\}(s) = 2e^{-2s}$

6

$\mathcal{L}\{\int_0^t (t-\tau)\sin(3\tau)d\tau\}(s) = \mathcal{L}\{t\}(s)\,\mathcal{L}\{\sin(3t)\}(s) = \frac{1}{s^2} \times \frac{3}{s^2+3^2}$

$\mathcal{L}\{f(t)\}(s) = \frac{2}{(s-1)^2+4} + 2e^{-2s} + \frac{3}{s^2(s^2+9)}$

b. $\frac{1}{(s+1)(s-2)} = -\frac{1}{3}\frac{1}{s+1} + \frac{1}{3}\frac{1}{s-2}$

$\mathcal{L}^{-1}\{\frac{1}{(s+1)(s-2)}\}(t) = \mathcal{L}^{-1}\{-\frac{1}{3}\frac{1}{s+1}\}(t) + \mathcal{L}^{-1}\{\frac{1}{3}\frac{1}{s-2}\}(t) = -\frac{1}{3}e^{-t} + \frac{1}{3}e^{2t}$

6

$\frac{e^{-s}}{(s-3)^4} = \frac{1}{3!}e^{-s}\frac{3!}{(s-3)^4} = \frac{1}{3!}e^{-s}G(s) \Rightarrow \mathcal{L}^{-1}\{\frac{1}{3!}e^{-s}G(s)\} = \frac{1}{3!}u_1(t)g(t-1)$

$g(t) = t^3 e^{3t}$

$\mathcal{L}^{-1}\{F(s)\}(t) = -\frac{1}{3}e^{-t} + \frac{1}{3}e^{2t} + \frac{1}{6}u_1(t)(t-1)^3 e^{3(t-1)}$

4. Consider the matrix $\mathbf{B} = \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix}$.

a. (6 points) Find the real-valued general solution of the differential system $\mathbf{x}' = \mathbf{B}\mathbf{x}$.

b. (6 points) Using method of variation of parameters, find the solution of the system

$$\mathbf{x}' = \mathbf{B}\mathbf{x} + \begin{pmatrix} e^t \\ 0 \end{pmatrix} \quad \text{with} \quad \mathbf{x}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

a. $\det B = 1-9 = -8 \neq 0 \qquad tr(B) = 1+1 = 2$

$\lambda^2 - 2\lambda - 8 = 0$ characteristic equation $\qquad \Delta = 4+32 = 36 = 6^2$

$\lambda_1 = \frac{2-6}{2} = -2 \qquad \lambda_2 = \frac{2+6}{2} = 4$ eigenvalues

6

$\lambda_1 = -2: \quad \begin{pmatrix} 3 & 3 \\ 3 & 3 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = 0 \iff a+b=0 \qquad v_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ eigenvector

$\lambda_2 = 4: \quad \begin{pmatrix} -3 & 3 \\ 3 & -3 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = 0 \iff a-b=0 \qquad v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

general solution of $x' = Bx$: $\boxed{c_1 e^{-2t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 e^{4t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}}$ c_1, c_2 arbitrary constants

b. Fundamental matrix $X(t) = \begin{pmatrix} e^{-2t} & e^{4t} \\ -e^{-2t} & e^{4t} \end{pmatrix}$

$\det X(t) = e^{2t} + e^{2t} = 2e^{2t} \neq 0$

$X^{-1}(t) = \begin{pmatrix} \frac{e^{2t}}{2} & -\frac{e^{2t}}{2} \\ \frac{e^{-4t}}{2} & \frac{e^{-4t}}{2} \end{pmatrix} \qquad X^{-1}(t)\, g(t) = \begin{pmatrix} \frac{e^{3t}}{2} \\ \frac{e^{-3t}}{2} \end{pmatrix}$

6

$\int_0^t X^{-1}(t) g(t)\, dt = \begin{pmatrix} \frac{e^{3t}}{6} - \frac{1}{6} \\ -\frac{e^{-3t}}{6} + \frac{1}{6} \end{pmatrix} \qquad X^{-1}(0)\, X_0 = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$

$X(t)\int_0^t X^{-1}(t) g(t)\, dt = \begin{pmatrix} -\frac{e^{-2t}}{6} + \frac{e^{4t}}{6} \\ -\frac{e^{t}}{3} + \frac{e^{-2t}}{6} + \frac{e^{4t}}{6} \end{pmatrix} \qquad X(t)\, X^{-1}(0) X_0 = \begin{pmatrix} \frac{1}{2}e^{-2t} + \frac{1}{2}e^{4t} \\ -\frac{1}{2}e^{-2t} + \frac{1}{2}e^{4t} \end{pmatrix}$

$$\boxed{X(t) = \begin{pmatrix} \frac{1}{3}e^{-2t} + \frac{2}{3}e^{4t} \\ -\frac{e^t}{3} - \frac{1}{3}e^{-2t} + \frac{2}{3}e^{4t} \end{pmatrix}}$$

5. Consider the initial value problem

$$y' - \frac{2}{t}y = -\frac{3}{t^2}$$
$$y(1) = 2$$

a. (5 points) Using the method of the integrating factor, find the exact solution of the initial value problem.

b. (4 points) Use Euler's method with a constant step size $h = 0.25$ to find approximate values of the solutions at time $t = 1.25, 1.5, 1.75$.

c. (4 points) Compute the local truncation errors at $t = 1, 1.25, 1.5, 1.75$.

d. (1 point) Is Euler approximation fair enough in this case?

a. Integrating factor $\frac{\mu'}{\mu} = -\frac{2}{t} \Rightarrow (\ln \mu)' = (-2\ln(t))' = \left(\ln\left(\frac{1}{t^2}\right)\right)'$

5

$\boxed{\mu(t) = \frac{1}{t^2}} \Rightarrow y(t) = \frac{1}{\mu(t)}\left[\int \mu(t)\, h(t)\, dt + C\right] =$

$= t^2\left[\int \frac{1}{t^2} \times \left(-\frac{3}{t^2}\right) dt + C\right] = \frac{1}{t} + ct^2$

$y(1) = 2 \Rightarrow 1 + C = 2 \Leftrightarrow \boxed{C = 1}$

$\boxed{y(t) = \frac{1}{t} + t^2}$

b. $y_1 = y(1) = \boxed{2}$ $\quad f(t,y) = -\frac{3}{t^2} + \frac{2}{t}y$

$y_2 = y_1 + f(t_1, y_1) \times h = 2 + \left(-\frac{3}{1^2} + \frac{2}{1}2\right) \times 0.25 = 2 + (-3+4) \times 0.25 = \boxed{2.25}$

4

$y_3 = y_2 + f(t_2, y_2) \times h = 2.25 + \left(-\frac{3}{1.25^2} + \frac{2}{1.25} \times 2.25\right) \times 0.25 = \boxed{2.67}$

$y_4 = y_3 + f(t_3, y_3) \times h = 2.67 + \left(-\frac{3}{1.5^2} + \frac{2}{1.5} \times 2.67\right) \times 0.25 \simeq \boxed{3.004}$

c. $\boxed{e_1 = 0}$ $\qquad$ d. No $\quad$ 1

$e_2 = y(1.25) - y_2 = \boxed{\frac{9}{80}} \simeq 0.11$

4

$e_3 = y(1.5) - y_3 \simeq \boxed{0{,}24}$

$e_4 = y(1.75) - y_4 \simeq \boxed{0.62}$

6. Consider the following differential equation

$$\frac{dy}{dt} = f(y) \quad \text{where } f(y) = (y^2 - 4)(y + 1)$$

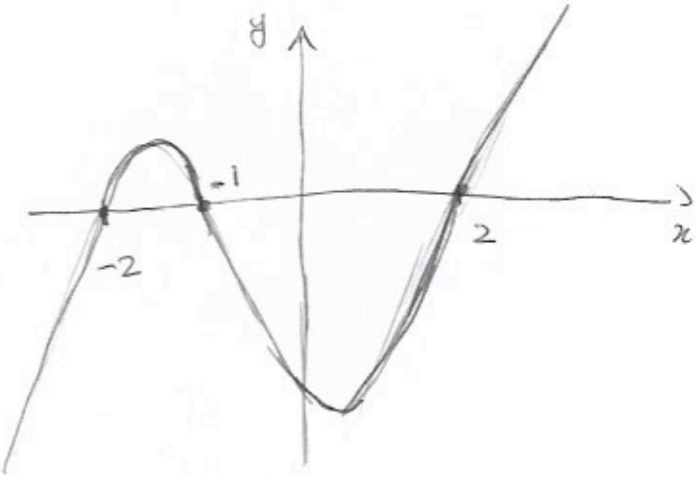
a. (2 points) Sketch the graph of $f(y)$ versus y.

b. (3 points) Find the critical points of the differential equation.

c. (5 points) **Without solving** the differential equation, determine whether each critical point is asymptotically stable or unstable.

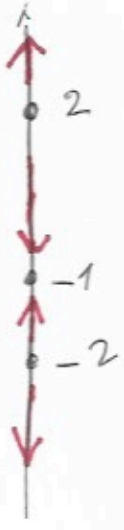
a.

2



3 b. $y = -1$, $y = -2$, $y = 2$

c.



asymptotically unstable

asymptotically stable

5

asymptotically unstable

7. For each of the following initial value problems (a) and (b),

(i) (1 point) Identify the equation (linear, nonlinear, separable, exact, homogeneous).

(ii) (6 points) Find the solution (you may leave your answer in an implicit form).

(a) $4\dfrac{dy}{dx} = \dfrac{3x^2}{y^3(1+x^3)}$ with $y(1) = 0$.

(b) $t\dfrac{dy}{dt} + y = 2e^t - t$ with $y(1) = -1$, $t > 0$.

1 (a) (i) $\boxed{\text{DE is separable non linear}}$

(ii) $4y^3\,dy = \dfrac{3x^2}{1+x^3}\,dx \iff y^4 = \ln|1+x^3| + c$

6

$y(1) = 0 \iff 0 = \ln(2) + c \iff c = -\ln(2)$

$$\boxed{y^4 = \ln|1+x^3| - \ln(2)}$$

1. (b) (i) DE is linear of order 1 non homogeneous.

(ii) $ty' + y = 2e^t - t \iff y' + \frac{1}{t}y = \frac{2e^t}{t} - 1$

$\iff \underbrace{uy' + \frac{1}{t}uy}_{(uy)' \text{ if } u' = \frac{u}{t}} = u\left(\frac{2e^t}{t} - 1\right)$

$(uy)'$ if $u' = \frac{u}{t} \iff \ln|u| = \ln|t| + \text{constant}$

So $u = t$

Now $ty' + y = 2e^t - t \iff uy = \int (2e^t - t)\,dt$

6

$\iff ty = 2e^t - \frac{t^2}{2} + C$

$y(1) = -1 \iff -1 = 2e - \frac{1}{2} + C \iff \boxed{C = -2e - \frac{1}{2}}$

$$\boxed{y(t) = \frac{1}{t}\left(2e^t - \frac{t^2}{2} - 2e - \frac{1}{2}\right)}$$

9

8. Consider the following differential system

$$\frac{dx}{dt} = x(2-x-y)$$

$$\frac{dy}{dt} = y(1-y)$$

a. (4 points) Find the critical points of the system.

b. (2 points) Write the Jacobian matrix.

c. (8 points) For each critical point find the corresponding linear system.

d. (2 points) Classify the critical point (1, 1) as nodal, spiral, center, ..., and determine whether it is asymptotically stable, stable or unstable. **No need to classify the other critical points.**

a. Solve $\begin{cases} x(2-x-y)=0 \\ y(1-y)=0 \end{cases}$

4

$y=0 \Rightarrow x(2-x)=0 \Rightarrow x=2,\ x=0$

$y=1 \Rightarrow x(1-x)=0 \Rightarrow x=0,\ x=1$

critical points are:

$\boxed{(0,0),(2,0),(0,1),(1,1)}$

b. $F(x,y)=x(2-x-y)$ $\quad G(x,y)=y(1-y)$

$F_x = 2-x-y-x = 2-2x-y$ $\quad F_y = -x$

2

$G_x = 0$ $\quad G_y = 1-y-y = 1-2y$

$$\boxed{J = \begin{pmatrix} 2-2x-y & -x \\ 0 & 1-2y \end{pmatrix}}$$

c. $(0,0)$

$$\boxed{\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}}$$

$(2,0)$

$$\boxed{\frac{d}{dt}\begin{pmatrix} x-2 \\ y \end{pmatrix} = \begin{pmatrix} -2 & -2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x-2 \\ y \end{pmatrix}}$$

8

$(0,1)$

$$\boxed{\frac{d}{dt}\begin{pmatrix} x \\ y-1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} x \\ y-1 \end{pmatrix}}$$

$(1,1)$

$$\boxed{\frac{d}{dt}\begin{pmatrix} x-1 \\ y-1 \end{pmatrix} = \begin{pmatrix} -1 & -1 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} x-1 \\ y-1 \end{pmatrix}}$$

2 d. $\lambda_1 = \lambda_2 = -1 \Rightarrow (1,1)$ $\boxed{\text{node}}$ for linear, $\boxed{\text{node or spiral point}}$ for non linear

$\boxed{\text{asymptotically stable}}$ for both systems