Final Review Problems

Exercise 1 Consider the differential equation

$$\frac{dy}{dt} = f(y) \quad \text{with} \quad f(y) = (y^2 - 3)(y + 4)$$

1. Sketch the graph of $f(y)$ versus y.
2. Determine the critical points (or equilibrium solutions) of the differential equation.
3. Classify each critical point as asymptotically stable, unstable or semistable. Draw the phase line and sketch several graphs of solutions in the ty-plane.

1. $f(y) = (y - \sqrt{3})(y + \sqrt{3})(y + 4)$

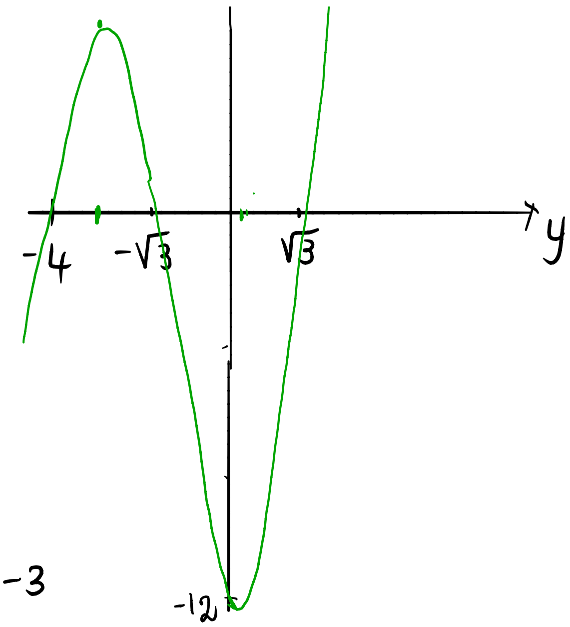
$f(y)$ is a cubic intersecting the y-axis at $y = \pm\sqrt{3}$ and $y = -4$

$f(0) = -12$

$f'(y) = (y^3 + 4y^2 - 3y - 12)'$
$= 3y^2 + 8y - 3$

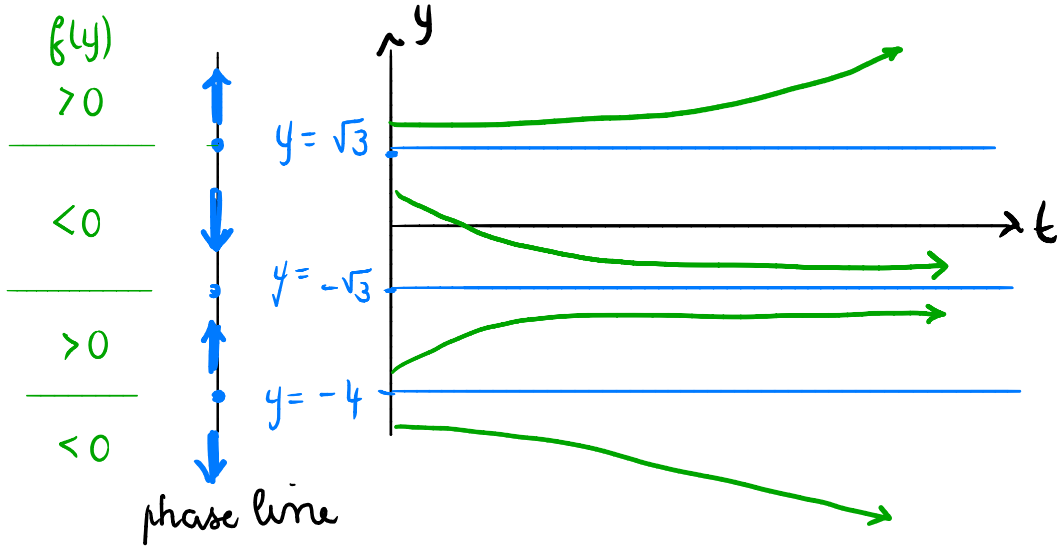
$f'(y) = 0 \qquad y = \dfrac{-4 \pm \sqrt{16 + 9}}{3} = \dfrac{-4 \pm 5}{3} = \left\langle \begin{array}{l} -3 \\ \frac{1}{3} \end{array} \right.$

$f(-3) = 6$, $f\left(\frac{1}{3}\right) = \left(\frac{1}{9} - 3\right)\left(\frac{1}{3} + 4\right) = -\frac{26}{9} \cdot \frac{13}{3} = -\frac{26}{27} \cdot 13 \sim -12.5$



2. The critical points of the DE are the constant solutions, such that $y' = 0$, i.e. $f(y) = 0$. They are $y = -4$, $y = -\sqrt{3}$ and $y = \sqrt{3}$

3.



$y = \sqrt{3}$ is unstable

$y = -\sqrt{3}$ is asymptotically stable

$y = -4$ is unstable

Exercise 2 **1.** State if the following differential equations are linear or nonlinear and solve the given initial value problems:

(1) $y' = 2y^2 + 2ty^2$ with initial condition $y(0) = 1$.

(2) $y' = 2y - e^{3t} + t$ with initial condition $y(0) = 1$.

2. For each of the two IVP, determine the largest interval containing $t = 0$ on which the solution is definied.

I. (1) $y' = 2(1+t)y^2$ 1rst order nonlinear DE, separable

$y=0$ is a solution; but it does not satisfy the initial condition.

So, can suppose $y \neq 0$ and divide both sides of the equation by y:

$\frac{1}{y^2} y' = 2(1+t)$. Integrate wrt to t:

$$\int \frac{1}{y^2} \underbrace{y' \, dt}_{dy} = 2\int (1+t)\, dt$$

i.e. $-\frac{1}{y} = 2t + t^2 + C_1$, C_1 constant

i.e. $y = -\frac{1}{2t+t^2+C}$

Determine C from the initial condition: $y(0)=1$

$1 = y(0) = -\frac{1}{C}$, i.e. $C = -1$,

The solution of the IVP is $y(t) = -\frac{1}{2t+t^2-1}$.

II. (1)

$2t+t^2-1 = 0 \iff t = 1 \pm \sqrt{1+1} = 1 \pm \sqrt{2}$

Since $\sqrt{2} > 1$, the largest interval containing 0 where the solution is defined is $]1-\sqrt{2}, 1+\sqrt{2}[$

(2) $y' = ty + (t-1)e^{t-\frac{1}{2}}$ is a 1st order nonhomogenious linear DE

I. Write it as $y' + p(t)y = q(t)$ where $p(t) = -t$, $q(t) = (t-1)e^{t-\frac{1}{2}}$

An integrating factor $\mu(t)$ is determined (up to an additive constant) by

$$\mu(t) = e^{\int p(t)dt} = e^{-\int t dt} = e^{-\frac{1}{2}t^2} + C$$

So we fix $\mu(t) = e^{-\frac{1}{2}t^2}$. Since

$$\mu(t)y' - \mu(t)t\,y = (\mu(t)\,y(t))'$$

the DE becomes

$$(\mu(t)\,y(t))' = \mu(t)\,g(t)$$

i.e. $\mu(t)\,y(t) = \int \mu(t) g(t)dt = \int e^{-\frac{1}{2}t^2}(t-1)\,e^{t-\frac{1}{2}}\,dt$

$$= \int e^{-\frac{1}{2}(t^2-2t+1)}(t-1)\,dt$$

$$= \int e^{-\frac{1}{2}(t-1)^2}\cdot(t-1)\,dt \qquad u = \tfrac{1}{2}(t-1)^2$$

$$du = (t-1)dt$$

$$= \int e^{-u}\,du$$

$$= -e^{-u} + C$$

$$= -e^{-\frac{1}{2}(t-1)^2} + C$$

i.e. $y(t) = -e^{-\frac{1}{2}(t-1)^2 + \frac{1}{2}t^2} + C \qquad [\,(t-1)^2 + t^2 = -2t+1\,]$

$$= -e^{t-\frac{1}{2}} + C$$

We fix the constant C using the initial condition $y(0)=1$:

$1 = y(0) = -e^{-1/2} + C$, so $C = 1 + e^{-1/2}$

Thus: the solution of the given IVP is $y(t) = -e^{t-\frac{1}{2}} + 1 + e^{-\frac{1}{2}}$

II. The solution of (1) is defined for all $t \in \mathbb{R}$, one can check it from its formula, but we could have also predicted it a priori using the theorem of existence and uniqueness of the solution of linear DE, since the functions p and g are continuous on the entire $\mathbb{R}$.

Exercise 3 Consider the differential equations

$$\text{a)}\qquad 3x^2y^2\,y' = -(1+2xy^3)$$
$$\text{b)}\qquad (1+2xy^3)y' = -3x^2y^2\,.$$

1. One of the two differential equations is exact. Which one?
2. Solve the differential equation that is exact.

1. $M(x,y) + N(x,y)\,y' = 0$ is exact provided $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$

[in this case: $M(x,y) = \dfrac{\partial \psi}{\partial x}$, $N(x,y) = \dfrac{\partial \psi}{\partial y}$]

- If $M(x,y) = 3x^2y^2$ and $N(x,y) = 1+2xy^3$, then

$\dfrac{\partial M}{\partial y} = 6x^2y$, $\dfrac{\partial N}{\partial y} = 6xy^2$. So $\dfrac{\partial M}{\partial y} \neq \dfrac{\partial N}{\partial x}$ and (a) is not exact.

- If $M(x,y) = 1+2xy^3$ and $N(x,y) = 3x^2y^2$, then

$\dfrac{\partial M}{\partial y} = 6xy^2$ and $\dfrac{\partial N}{\partial x} = 6xy^2$, so $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$ and (b) is exact.

2. If $\psi(x,y)$ satisfies $M(x,y) = \dfrac{\partial \psi}{\partial x}$, $N(x,y) = \dfrac{\partial \psi}{\partial y}$, then

$$\psi(x,y) = \int M(x,y)\,dx + h(y) = \int (1+2xy^3)\,dx + h(y)$$
$$= x + x^2y^3 + h(y)$$

$3x^2y^2 = N(x,y) = \dfrac{\partial \psi}{\partial y} = 3x^2y^2 + h'(y) \Rightarrow h'(y) = 0 \Rightarrow h(y) = C_0$ constant

Thus $\psi(x,y) = x + x^2y^3 + C_0$

The solutions of the differential equation (b) satisfy the implicit equation

$$x + x^2y^3 = c \quad , \quad c \text{ constant}$$

Solutions can be found in explicit form; for $x \neq 0$

$$y = \left(\frac{c-x}{x^2}\right)^{1/3} = \left(\frac{c}{x^2} - \frac{1}{x}\right)^{1/3}$$

Exercise 4 Consider the system of linear differential equations $\mathbf{x}' = \mathbf{A}\mathbf{x}$, where $\mathbf{A}$ is one of the following two matrices:

$$(a)\ \begin{pmatrix} -4 & -2 \\ 3 & -11 \end{pmatrix}; \qquad (b)\ \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}$$

For each of them:

1. Find the general solution of the linear system.
2. Determine the equilibrium solutions, their type and stability.
3. Solve the IVP with the inital condition $\mathbf{x}(0) = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$.

(a) $\det(A - \lambda I) = \begin{vmatrix} -4-\lambda & -2 \\ 3 & -11-\lambda \end{vmatrix} = (4+\lambda)(11+\lambda) + 6 = \lambda^2 + 15\lambda + 50$

(1) $\lambda^2 + 15\lambda + 50 = 0 \Leftrightarrow \lambda = \dfrac{-15 \pm \sqrt{225-200}}{2} = \dfrac{-15 \pm 5}{2} = \begin{cases} -10 \\ -5 \end{cases}$

A has eigenvalues $\lambda_1 = -10$ and $\lambda_2 = -20$

- $\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ is eigenvector of A for the eigenvalue $\lambda_1 = -10 \Leftrightarrow$ $(A - \lambda_1 I)\mathbf{v} = \mathbf{0}$, i.e. $\begin{cases} 6v_1 - 2v_2 = 0 \\ 3v_1 - v_2 = 0 \end{cases}$, i.e. $v_2 = 3v_1$. So $\begin{pmatrix} 1 \\ 3 \end{pmatrix}$ is eigenvector of A for $\lambda_1 = -10$.
- $\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ is eigenvector of A with eigenvalue $\lambda_2 = -5 \Leftrightarrow$ $(A - \lambda_2 I)\mathbf{v} = \mathbf{0}$, i.e $\begin{cases} v_1 - 2v_2 = 0 \\ 3v_1 - 6v_2 = 0 \end{cases} \Leftrightarrow v_1 = 2v_2$. So $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ is eigenvector for $\lambda_2 = -5$.

The general solution for the case (a) is

$\mathbf{x}(t) = c_1 e^{-10t}\begin{pmatrix} 1 \\ 3 \end{pmatrix} + c_2 e^{-5t}\begin{pmatrix} 2 \\ 1 \end{pmatrix}$, c_1, c_2 constant

(2) Since $\det(A) \neq 0$ (since $\lambda = 0$ is not an eigenvalue of A, or directly, since $\det(A) = 50 \neq 0$), the unique equilibrium solution is $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$. Since A has two real, distinct and negative eigenvalues, the critical point is a nodal sink and is

asymptotically stable.

(3) $\begin{pmatrix}-1\\2\end{pmatrix} = \mathbf{x}(0) = C_1\begin{pmatrix}1\\3\end{pmatrix} + C_2\begin{pmatrix}2\\1\end{pmatrix} \Leftrightarrow \begin{cases} C_1 + 2C_2 = -1 \\ 3C_1 + C_2 = 2 \end{cases}$, i.e. $\begin{cases} C_1 = 1 \\ C_2 = -1 \end{cases}$

The solution of the IVP is $\mathbf{x}(t) = e^{-10t}\begin{pmatrix}1\\3\end{pmatrix} - e^{-5t}\begin{pmatrix}2\\1\end{pmatrix}$

(i.e. $\begin{cases} x_1(t) = e^{-10t} - 2e^{-5t} \\ x_2(t) = 3e^{-10t} - e^{-5t} \end{cases}$

(iv) In this case, A is a diagonal matrix. Its eigenvalues are $\lambda_1 = \lambda_2 = -2$. There are two linearly independent (i.e. not proportional) eigenvectors, namely $\begin{pmatrix}1\\0\end{pmatrix}$ and $\begin{pmatrix}0\\1\end{pmatrix}$

(1) The general solution of the DE is

$$\mathbf{x}(t) = C_1 e^{-2t}\begin{pmatrix}1\\0\end{pmatrix} + C_2 e^{-2t}\begin{pmatrix}0\\1\end{pmatrix}$$

(2) The unique critical solution is $(0,0)$ because $\det(A) \neq 0$.
There are two equal eigenvalues strictly negative eigenvalues with two lin. indep. eigenvectors. So $(0,0)$ is a stable proper node and it is asymptotically stable

(3) $\begin{pmatrix}-1\\2\end{pmatrix} = \mathbf{x}(0) = C_1\begin{pmatrix}1\\0\end{pmatrix} + C_2\begin{pmatrix}0\\1\end{pmatrix} \Rightarrow \begin{cases} C_1 = -1 \\ C_2 = 2 \end{cases}$. The solution to the IVP is

$\mathbf{x}(t) = -e^{-2t}\begin{pmatrix}1\\0\end{pmatrix} + 2e^{-2t}\begin{pmatrix}0\\1\end{pmatrix}$, i.e. $\begin{cases} x_1(t) = -e^{-2t} \\ x_2(t) = 2e^{-2t} \end{cases}$

Exercise 6 If the Wronskian of f and g is $t\cos t - \sin t$, and if $u = f + 3g$, $v = f - g$, find the Wronskian of u and v.

$$W[u,v](x) = \begin{vmatrix} u & v \\ u' & v' \end{vmatrix} = uv' - u'v = (f+3g)(f-g)' - (f+3g)'(f-g)$$

$$= (f+3g)(f'-g') - (f'+3g')(f-g)$$

$$= \cancel{ff'} + 3gf' - fg' - \cancel{3gg'} - \cancel{f'f} - 3g'f + f'g + \cancel{3g'g}$$

$$= -4fg' + 4f'g$$

$$= -4(fg' - f'g) = -4\begin{vmatrix} f & g \\ f' & g' \end{vmatrix} = -4W[f,g] = -4(t\cos t - \sin t)$$

Exercise 5 A mass of 2 kilograms is attached to a spring with spring constant 26 N/m. The damping constant γ for the system is 12 N.sec/m. Suppose the mass is moved 0.5 m upward of equilibrium and given an initial upward velocity of 3 m/sec.

1. Determine the initial value problem describing the movement of the mass.
2. Find the position of the mass at any time t.
3. Find the amplitude, the quasi-frequency and the quasi-period of the solution.

1. $y(t)$ = position of the mass at time t (measured along a vertical y-axis with positive downward direction and $y=0$ at equilibrium)

The DE of the motion is $my'' + \gamma y' + ky = 0$, i.e.

$$2y'' + 12y' + 26y = 0$$

with initial conditions $y(0) = -0.5$, $y'(0) = -3$ m/sec.

2. $y'' + 6y' + 13y = 0$, $y(0) = -0.5$, $y'(0) = -3$

Homogeneous linear DE with constant coeffs.

Characteristic equation: $\lambda^2 + 6\lambda + 13 = 0$, $\lambda = -3 \pm \sqrt{9-13} = -3 \pm 2i$

General solution of the DE: $y(t) = C_1 e^{-3t}\cos(2t) + C_2 e^{-3t}\sin(2t)$, C_1, C_2 const

Determine C_1, C_2 using the initial conditions:

$$\begin{cases} -0.5 = y(0) = C_1 \\ -3 = y'(0) = -3C_1 + 2C_2 \end{cases}$$

$\left[y'(t) = -3C_1 e^{-3t}\cos(2t) - 2C_1 e^{-3t}\sin(2t) - 3C_1 e^{-3t}\sin(2t) + 2C_2 e^{-3t}\cos(2t) \right]$

$$\Rightarrow \begin{cases} C_1 = -0.5 \\ C_2 = (-3 - 1.5)/2 = -4.5/2 = -2.25 \end{cases}$$

The position at time t is $y(t) = -0.5e^{-3t}\cos(2t) - 2.25e^{-3t}\sin(2t)$

$$= -\frac{1}{2}e^{-3t}\left(\cos(2t) + \frac{9}{2}\sin(2t)\right)$$

3. Want to write $y(t) = e^{-3t} R\cos(2t - \delta)$ where

$$R\cos(2t - \delta) = -\frac{1}{2}\cos(2t) - \frac{9}{4}\sin(2t)$$

$R\cos(2t-\delta) = R\cos(2t)\cos\delta - R\sin(2t)\sin\delta$. So look for R, δ so that

$$\begin{cases} R\cos\delta = -\frac{1}{2} \\ R\sin\delta = -\frac{9}{4} \end{cases} \Rightarrow R^2 = \frac{1}{4} + \frac{81}{16} = \frac{85}{16} \Rightarrow R = \frac{\sqrt{85}}{4}$$

- amplitude $= e^{-3t}R = e^{-3t}\frac{\sqrt{85}}{4}$
- quasi-frequency $= 2$
- quasi-period $= \frac{2\pi}{2} = 2$

Exercise 7 Consider the differential equation $5y'' - 4y' = 0$.

1. Write a system of first order differential equations which is equivalent to the given equation.
2. Determine the general solution of the system in **1**.
3. Determine the general solution of the second order differential equation.
4. What can you say about the critical points?
5. Determine a suitable form for the particular solution $Y(t)$ of the differential equation

$$5y'' - 4y' = t^2 + e^{-4/5t}$$

if the method of undetermined coefficients is to be used. You do not need to find the value of the coefficients or solve the differential equation.

(1) Set $x_1 = y$, $x_2 = y'$. Hence the DE is equivalent to $\begin{cases} x_1' = x_2 \\ x_2' = 4/5\, x_2 \end{cases}$ or

$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = A\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ with $A = \begin{pmatrix} 0 & 1 \\ 0 & 4/5 \end{pmatrix}$

(2) $\det(A - \lambda I) = \begin{vmatrix} -\lambda & 1 \\ 0 & 4/5 - \lambda \end{vmatrix} = \lambda(\lambda - 4/5) = 0$, i.e. the eigenvalues of A are $\lambda_1 = 0$ and $\lambda_2 = 4/5$.

$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is eigenvector of $\lambda_1 = 0$ and $\begin{pmatrix} 5 \\ 4 \end{pmatrix}$ is eigenvector for $\lambda_2 = \frac{4}{5}$ [because $A - \frac{4}{5}I = \begin{pmatrix} -4/5 & 1 \\ 0 & 0 \end{pmatrix}$, and $-\frac{4}{5}v_1 + v_2 = 0$ gives $v_2 = \frac{4}{5}v_1$]

The general solution of the system is $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = C_1\begin{pmatrix} 1 \\ 0 \end{pmatrix} + C_2 e^{-4/5t}\begin{pmatrix} 5 \\ 4 \end{pmatrix}$

(3) The general solution of the 2nd order equation is

$$y(t) = x_1(t) = C_1 + 5C_2 e^{-4/5t}$$

(4) Every point of the form $\begin{pmatrix} x_1 \\ 0 \end{pmatrix}$ is a critical point

Every solution tends to a critical point as $t \to +\infty$ because

$\lim_{t \to +\infty} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = C_1\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ for every solution $\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$ as in (2)

(5) Since $e^{-5/4t}$ is an exponential function which is a solution of the homogeneous DE $5y'' - 4y' = 0$, whereas $t^2 e^{5/4t}$ is not, and since t^2 is a polynomial of degree 2 which is not a solution of $5y'' - 4y' = 0$, where any constant function is a solution, we can look for $Y(t)$ of the form

$$Y(t) = Ate^{-5/4t} + Bt^2 + Ct$$

Exercise 8 **1.** Find the Laplace transform of

$$f(t) = -2e^{3t}\cos(t) - \ln(t^2+1)\delta(t-3) + \int_0^t \tau^2 \sin(3(t-\tau))d\tau$$

2. Find the inverse Laplace transform of

$$F(s) = \frac{2}{(s-3)(s-2)(s-1)} - 3\frac{e^{-5s}}{(s+6)^8}.$$

1. By linearity, for all $s>0$ where all the Laplace transforms are def,

$$\mathcal{L}\{f\}(s) = -2\,\mathcal{L}\{e^{3t}\cos t\}(s) - \mathcal{L}\{\ln(t^2+1)\,\delta(t-3)\}(s) + \mathcal{L}\{t^2 * \sin(3t)\}(s)$$

$$= -2\,\mathcal{L}\{\cos t\}(s-3) - e^{-3s}\ln(3^2+1) + \mathcal{L}\{t^2\}(s)\,\mathcal{L}\{\sin(3t)\}(s)$$

$$= -2\frac{s-3}{(s-3)^2+1} - \ln 10 \cdot e^{-3s} + \frac{2}{s^3}\,\frac{1}{s^2+9}$$

$\mathcal{L}\{f\}(s)$ is defined for $s>3$

(because $\mathcal{L}\{\cos t\}(s-3)$ is defined for $s-3>0$ and all the other Laplace transforms are def for $s>0$)

2. Partial fraction decomposition gives

$$\frac{2}{(s-3)(s-2)(s-1)} = \frac{A}{s-3} + \frac{B}{s-2} + \frac{C}{s-1} = \frac{A(s^2-3s+2)+B(s^2-4s+3)+C(s^2-5s+6)}{(s-3)(s-2)(s-1)}$$

i.e. $(A+B+C)s^2 + (-3A-4B-5C)s + (2A+3B+6C) = 2$

This is a polynomial identity, It holds if and only if

$$\begin{cases} A+B+C=0 \\ 3A+4B+5C=0 \\ 2A+3B+6C=2 \end{cases} \Rightarrow \begin{cases} A+B+C=0 \\ B+2C=0 \\ A+B-C=-2 \end{cases} \Rightarrow \begin{cases} C=1 \\ B=-2 \\ A=1 \end{cases}$$

Hence by linearity of $\mathcal{L}^{-1}$:

$$\mathcal{L}^{-1}\{F\}(t) = \mathcal{L}^{-1}\left\{\frac{1}{s-3}\right\}(t) - 2\,\mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\}(t) + \mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\}(t)$$

$$- 3\,u_5(t)\,\mathcal{L}^{-1}\left\{\frac{1}{(s+6)^8}\right\}(t-5)$$

$$= e^{3t} - 2e^{2t} + e^{t} - \frac{3}{7!}u_5(t)(t-5)^7 e^{-6(t-5)}$$

Exercise 9 Consider the piecewise defined function

$$g(t) = \begin{cases} 1 & \text{if } 0 \le t < 2 \\ t-2 & \text{if } 2 \le t \end{cases}$$

1. Express $g(t)$ in terms of unit step functions.

2. Find the solution of the initial value problem

$$y'' + y = g(t) \quad \text{with} \quad y(0) = 0 \text{ and } y'(0) = 0$$

1. $g(t) = u_0(t) - u_2(t) + (t-2)u_2(t) = u_0(t) + (t-3)u_2(t)$

2. Apply the Laplace transform to both sides of the DE and set $Y = \mathcal{L}\{y\}$.
By linearity of the Laplace transform, we have

$$\mathcal{L}\{y''\}(s) + \mathcal{L}\{y\}(s) = \mathcal{L}\{g\}(s)$$

$$s^2 Y(s) - s\underbrace{y(0)}_{=0} - \underbrace{y'(0)}_{=0} + Y(s) = \mathcal{L}\{g\}(s)$$

Compute $\mathcal{L}\{g\}(s)$: where $g_1(t-2) = t-3$, i.e. $g_1(t) = t-1$

$$\mathcal{L}\{g\}(s) = \mathcal{L}\{1\}(s) + \mathcal{L}\{g_1(t-2)u_2(t)\}(s)$$

$$\underset{(s>0)}{=} \frac{1}{s} + e^{-2s}\mathcal{L}\{t-1\}(s) = \frac{1}{s} + e^{-2s}\left(\frac{1}{s^2} - \frac{1}{s}\right)$$

$$\boxed{\mathcal{L}\{f(t-c)u_c(t)\}(s) = \mathcal{L}\{f\}(s)e^{-st}}$$

So $(1+s^2)Y(s) = \frac{1}{s} + \left(\frac{1}{s^2} - \frac{1}{s}\right)e^{-2s}$, i.e.

$$Y(s) = \frac{1}{s(1+s^2)} + \frac{1}{s^2(1+s^2)}e^{-2s} - \frac{1}{s(1+s^2)}e^{-2s}$$

$$\left[\frac{1}{s(1+s^2)} = \frac{1}{s} - \frac{s}{1+s^2} \qquad \frac{1}{s^2(1+s^2)} = \frac{1}{s^2} - \frac{1}{1+s^2} \right]$$

$$= \frac{1}{s} - \frac{s}{1+s^2} + \left(\frac{1}{s^2} - \frac{1}{1+s^2} - \frac{1}{s} + \frac{s}{1+s^2}\right)e^{-2s}$$

Hence

$$y(t) = \mathcal{L}^{-1}\{Y\}(t) = 1 - \cos t + (t - 2 - \sin(t-2) - 1 + \cos(t-2))u_2(t)$$

$$= 1 - \cos t + (t - 3 - \sin(t-2) + \cos(t-2))u_2(t)$$

Exercise 10 Consider the following system of differential equations:

$$\frac{dx}{dt} = y(3-x-y)$$
$$\frac{dy}{dt} = x(2-x)$$

1. Find all the critical points of the system.
2. Compute the Jacobian matrix for the system.
3. For each critical point, find the corresponding approximating linear system. Find the eigenvalues of each linear system and classify each critical point according to type (nodal, spiral, center,...) and stability (asymptotically stable, stable, or unstable).

1. The critical points are the solutions of the system $\begin{cases} F(x,y)=0 \\ G(x,y)=0 \end{cases}$ where $\begin{cases} F(x,y) = y(3-x-y) \\ G(x,y) = x(2-x) \end{cases}$

$\begin{cases} y(3-x-y)=0 \\ x(2-x)=0 \end{cases} < \begin{matrix} x=0 \\ x=2 \end{matrix}$, which gives as critical points $(0,0)$, $(2,0)$, $(0,3)$, $(2,1)$

2. $F_x = -y$, $F_y = 3-x-2y$, $G_x = 2-2x$, $G_y = 0$ $\Big\} \Rightarrow$ The Jacobian matrix is

$$J(x,y) = \begin{pmatrix} F_x & F_y \\ G_x & G_y \end{pmatrix} = \begin{pmatrix} -y & 3-x-2y \\ 2-2x & 0 \end{pmatrix}$$

3. $\boxed{(0,0)}$: $J(0,0) = \begin{pmatrix} 0 & 3 \\ 2 & 0 \end{pmatrix}$. So the linear approximation at $(0,0)$ is $\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & 3 \\ 2 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$. The eigenvalues are the solutions of $\det(J(0,0) - \lambda I) = \det\begin{pmatrix} -\lambda & 3 \\ 2 & -\lambda \end{pmatrix} = \lambda^2 - 6 = 0$, i.e. $\lambda = \pm\sqrt{6}$. So $(0,0)$ is a saddle point and is unstable (two real eigenvalues with opposite sign).

$\boxed{(2,0)}$: $J(2,0) = \begin{pmatrix} 0 & 1 \\ -2 & 0 \end{pmatrix}$. The linear approximation at $(2,0)$ is $\frac{d}{dt}\begin{pmatrix} u \\ w \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -2 & 0 \end{pmatrix}\begin{pmatrix} u \\ w \end{pmatrix}$ where $\begin{cases} u = x-2 \\ w = y \end{cases}$

The eigenvalues are the solutions of $\det(J(2,0)-\lambda I) = \det\begin{pmatrix}-\lambda & 1\\ -2 & -\lambda\end{pmatrix} = \lambda^2+2=0$, i.e. $\lambda = \pm i\sqrt{2}$. For this linear system, the critical point $(u,v)=(0,0)$ is a center and is stable. For the nonlinear system, $(2,0)$ is either a center or a spiral point; its stability is indetermined.

$\boxed{(0,3)}$: $J(0,3) = \begin{pmatrix}-3 & -3\\ 2 & 0\end{pmatrix}$. So the linear approximation at $(0,3)$ is $\frac{d}{dt}\begin{pmatrix}u\\w\end{pmatrix} = \begin{pmatrix}-3 & -3\\ 2 & 0\end{pmatrix}\begin{pmatrix}u\\w\end{pmatrix}$ where $\begin{cases}u=x\\ w=y-3\end{cases}$. The eigenvalues are solutions of $\det(J(0,3)-\lambda I) = \det\begin{pmatrix}-3-\lambda & -3\\ 2 & -\lambda\end{pmatrix} = \lambda(3+\lambda)+6 = \lambda^2+3\lambda+6=0$

i.e. $\lambda = \frac{3\pm\sqrt{9-24}}{2} = \frac{3\pm i\sqrt{15}}{2}$, so $(0,3)$ is a spiral point and it is unstable because the matrix of the linear approximation has two complex conjugate eigenvalues with real part >0.

$\boxed{(2,1)}$ $J(2,1) = \begin{pmatrix}-1 & -1\\ -2 & 0\end{pmatrix}$. The linear approximation at $(2,0)$ is

$\frac{d}{dt}\begin{pmatrix}u\\w\end{pmatrix} = \begin{pmatrix}-1 & -1\\ -2 & 0\end{pmatrix}\begin{pmatrix}u\\w\end{pmatrix}$ where $\begin{cases}u=x-2\\ w=y-1\end{cases}$. The eigenvalues are the solutions of $\det(J(2,1)-\lambda I) = \det\begin{pmatrix}-1-\lambda & -1\\ -2 & -\lambda\end{pmatrix} = \lambda(1+\lambda)-2 = \lambda^2+\lambda-2=0$

i.e. $\lambda = \frac{-1\pm\sqrt{9}}{2}$, i.e. $\lambda_1=1, \lambda_2=2$. The critical point is a saddle point and is unstable, as in the case of $(0,0)$