Practical information on the Final Exam

- The Final Exam on Friday, April 24, 1 pm–5:30 pm (Atlanta time).
  The 4 hours 30 minutes of the exam include an additional 1 hour and 40 minutes for scanning and sending me your file.
- The Final Exam will be released on Canvas $\longrightarrow$ Quizzes.
- Please email your scanned PDF solution by the deadline to:
  angela.pasquale@univ-lorraine.fr or angela.pasquale@georgiatech-metz.fr .
- Before submitting it, please check that your file is complete and readable.
- Final Exam coverage: the Final Exam covers most of the material from this semester. Details can be found on the Review Sheet.
- For the Final Exam, *you will need to know:* the material and the examples from the lectures; the exercises covered in the recitations. The solution sheets for the recitations made in March/April are available on this webpage. Please be sure to review them. Below you will find *several* additional review problems. You might want to try *some* of them *if* you have extra time. You can select those on the topics you feel you need extra practice.
- There will be about 8 problems in the Final Exam. The format will be similar (but not equal) to the Final Exam from the previous semester posted on the webpage of the class.
- As in the previous online quizzes and in Midterm 2, you can write your solution on the same pdf I sent you, for instance if you have a tablet or you can print it. If you need extra space, add as many pages as needed.
  Otherwise, please write your solutions on blank paper. You do not need to copy the questions, just clearly *mark the number of the exercises and their questions* and *separate the different exercises with a horizontal line.*
- Please call your file “yourname-Final".
- The Final Exam is a take-home exam. You can use scientific calculators, but you have to justify your answers and show your work.
- You will need the table of Laplace transform from your textbook. A copy will be provided for your convenience.
- In the Final Exam, you have to solve the problems by yourself. You are not allowed to discuss problems and solutions with other people in any form. Please abide by the Honor Code.
- I will be online during the whole exam time. You can send me messages by email. I will do my best to answer as soon as I can. Sometimes I will be answering to other people. So please be patient. Also, please understand that there are questions to which I cannot answer: for instance, if your solution is correct or not.

Final Review Problems

Exercise 1 Consider the differential equation

$$\frac{dy}{dt} = f(y) \quad \text{with} \quad f(y) = (y+3)(y^2-4)$$

1. Sketch the graph of $f(y)$ versus y.
2. Determine the critical points (or equilibrium solutions) of the differential equation.
3. Classify each critical point as asymptotically stable, unstable or semistable. Draw the phase line and sketch several graphs of solutions in the ty-plane.

Exercise 2 **1.** State if the following differential equations are linear or nonlinear and solve the given initial value problems:

(1) $y' = 2y^2 + 2ty^2$ with initial condition $y(0) = 1$.
(2) $y' = 2y - e^{3t} + t$ with initial condition $y(0) = 1$.

2. For each of the two IVP, determine the largest interval containing $t = 0$ on which the solution is definied.

Exercise 3 A tank initially contains 50 gallons of brine with 10 lb of salt dissolved into it. Brine containing 0.5 lb of salt per gallon flows from an outside source into the tank at a rate of 4 gal/min. The well-stirred mixture leaves the tank at the same rate of 4 gal/min.

1. Write the initial value problem describing the quantity $Q(t)$ of salt in the tank at every time $t \geq 0$.
2. Find the amount $Q(t)$ of salt in the tank at time $t \geq 0$.
3. Determine the limit of $Q(t)$ as $t \to \infty$.

Exercise 4 Consider the differential equations

$$\begin{array}{ll} \text{a)} & 3x^2y^2\,y' = -(1+2xy^3) \\ \text{b)} & (1+2xy^3)y' = -3x^2y^2\,. \end{array}$$

1. One of the two differential equations is exact. Which one?
2. Solve the differential equation that is exact.

Exercise 5 Consider the system of linear differential equations $\mathbf{x}' = \mathbf{A}\mathbf{x}$, where $\mathbf{A}$ is one of the following two matrices:

$$(a)\ \begin{pmatrix} -4 & -2 \\ 3 & -11 \end{pmatrix}; \qquad (b)\ \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}$$

For each of them:

1. Find the general solution of the linear system.
2. Determine the equilibrium solutions, their type and stability.
3. Solve the IVP with the inital condition $\mathbf{x}(0) = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$.

Exercise 6 If the Wronskian of f and g is $t\cos t - \sin t$, and if $u = f + 3g, v = f - g$, find the Wronskian of u and v.

Exercise 7 A mass of 2 kilograms is attached to a spring with spring constant 26 N/m. The damping constant γ for the system is 12 N.sec/m. Suppose the mass is moved 0.5 m upward of equilibrium and given an initial upward velocity of 3 m/sec.

1. Determine the initial value problem describing the movement of the mass.
2. Find the position of the mass at any time t.
3. Find the amplitude, the quasi-frequency and the quasi-period of the solution.

Exercise 8 Consider the differential equation $5y'' - 4y' = 0$.

1. Write a system of first order differential equations which is equivalent to the given equation.
2. Determine the general solution of the system in **1**.
3. Determine the general solution of the second order differential equation.
4. What can you say about the critical points?
5. Determine a suitable form for the particular solution $Y(t)$ of the differential equation

$$5y'' - 4y' = t^2 + e^{-4/5t}$$

   if the method of undetermined coefficients is to be used. You do not need to find the value of the coefficients or solve the differential equation.

Exercise 9 **1.** Find the Laplace transform $\mathcal{L}\{f\}(s)$ of the following functions. Indicate for which values of s it is defined.

   a) $f(t) = t^2 e^{-3t} + t^3 + 3$;
   b) $f(t) = -2e^{3t}\cos(2t) + t\cos(2t) - te^{3t}\cos(2t)$.

2. Find the inverse Laplace transform of

$$F(s) = \frac{2}{(s-3)(s-2)(s-1)} - 3\frac{e^{-5s}}{(s+6)^8}.$$

Exercise 10 Consider the piecewise defined function

$$g(t) = \begin{cases} 1 & \text{if } 0 \le t < 2 \\ t-2 & \text{if } 2 \le t \end{cases}$$

1. Express $g(t)$ in terms of unit step functions.
2. Find the solution of the initial value problem

$$y'' + y = g(t) \quad \text{with} \quad y(0) = 0 \text{ and } y'(0) = 0$$

Exercise 11 Consider the following system of differential equations:

$$\begin{aligned} \frac{dx}{dt} &= y(3-x-y) \\ \frac{dy}{dt} &= x(2-x) \end{aligned}$$

1. Find all the critical points of the system.
2. Compute the Jacobian matrix for the system.
3. For each critical point, find the corresponding approximating linear system. Find the eigenvalues of each linear system and classify each critical point according to type (nodal, spiral, center,...) and stability (asymptotically stable, stable, or unstable).