Georgia Tech – Lorraine
Fall 2019
Differential Equations
Math 2552
5/12/2019

Last Name:
First Name:

Final Exam

Total time: 2 hours and 50 minutes.
Total points: 100 points

Please organize your work clearly, neatly, and legibly.
Identify your answers.
Show your work and justify your answers.
If you need extra space, use the back sides of each page.
Please do not use red or pink ink.

Calculators, notes, cell phones, and books are not allowed.
A table of Laplace transforms is provided.

I wish you success in your exam.

EX	points	
1	10	
2	10	
3	12	
4	16	
5	12	
6	12	
7	12	
8	16	
TOT	100	

Exercise 1 (2+3+5=10 points)

Consider the differential equation $y' = f(y)$ where $f(y) = -y(y-1)^2$.

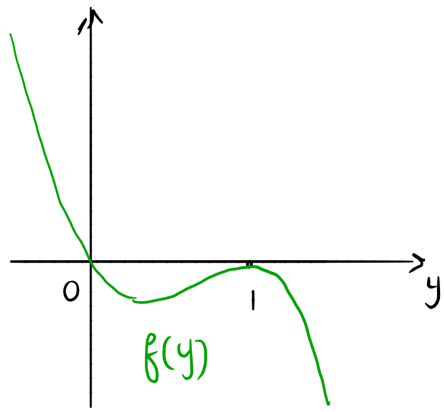
1. Sketch the graph of $f(y)$ versus y.
2. Determine the critical (or equilibrium) points.
3. Classify each critical point as asymptotically stable, unstable or semistable. Draw the phase line and sketch several graphs of solutions in the ty-plane.

1. $f(y) = -y(y-1)^2$ is a cubic function (polynomial of 3rd degree in y)

The cubic equation $f(y)=0$ has solutions $y=0$, $y=1$ (double)

They are the intersections of the graph of f with the y axis.

Since $(y-1)^2 \geq 0$ for all $y \in \mathbb{R}$, the sign of $f(y)$ is determined by the factor $-y$. Hence $f(y) > 0$ for $y<0$ and $f(y)<0$ for $y>0$



REM: $f(y) = -y(y-1)^2$ has a local minimum between 0 and 1. The value y at which it occurs can be found by solving $f'(y)=0$:

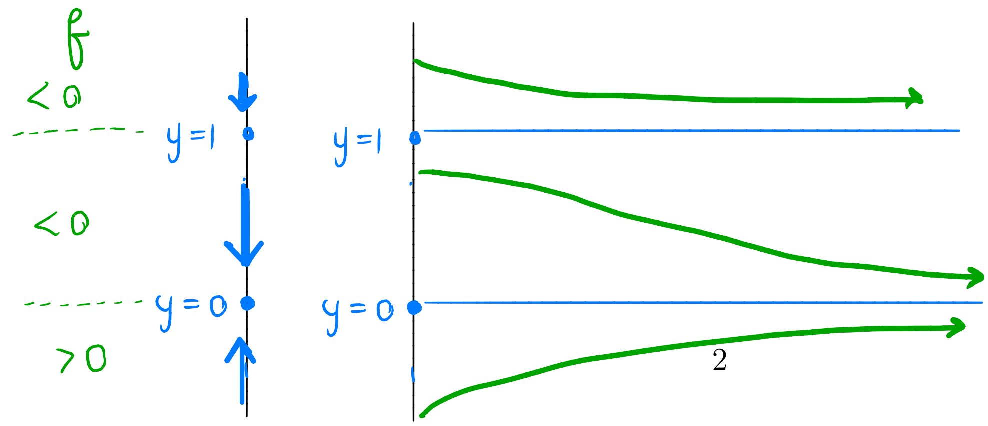
Namely: $0 = f'(y) = -(y-1)^2 - 2y(y-1)$
$= (y-1)[-y+1-2y]$
$= (y-1)(1-3y)$

gives $y=1$ and $y=1/3$. Thus the local min is at $y=1/3$, with $f(1/3) = -\frac{4}{27}$

2. Critical points = constant solutions of the DE
= solutions of $f(y)=0$

They are $y=0$ and $y=1$

3.



$y=1$ semistable

$y=0$ asymptotically stable

phase line

Exercise 2 (3+5+2=10 points)
A hot metal bar is placed in a room at a constant temperature of 20° C. After 6 minutes the temperature of the bar is measured as 80° C. Two minutes later, the temperature of the bar has decreasead to 50° C.
Suppose that Newton's law of cooling applies with transmission coefficient k.

1. Write an initial value problem modeling the temperature of the bar as a function of time.
2. Solve the initial value problem. The transmission coefficient k has to be computed. (Leave your answer in term of ln.)
3. What was the initial temperature of the metal bar?

1. $y(t)$ = temperature (in °C) of the metal bar at time t (in sec)
Then $\frac{dy}{dt} = -k(y-T)$, where $T = 20°C$
k = transmission coefficient

The required IVP is:

$$\boxed{\frac{dy}{dt} = -k(y-20) \text{ with } y(6)=80 \text{ and } y(8)=50}$$

2. Suppose $y \neq 20$ (which is not solution of the IVP). Then $\frac{1}{y-20}\frac{dy}{dt} = -k$
Integrate wrt t: $\int \frac{1}{y-20} \underbrace{\frac{dy}{dt} dt}_{dy} = -k \int dt$

$\ln|y-20| = -kt + C_0$, i.e. $|y-20| = e^{C_0} e^{-kt}$, i.e. $y = 20 + Ce^{-kt}$, C = constant

$$\begin{cases} y(t) = 20 + Ce^{-kt} \\ y(6) = 80 \end{cases}$$

$y(8)=50$ additional value allowing us to compute k

$$\begin{cases} 80 = 20 + ce^{-6k} \\ 50 = 20 + ce^{-8k} \end{cases} \Rightarrow \begin{cases} 60 = ce^{-6k} \\ 30 = ce^{-8k} \end{cases} \Rightarrow e^{2k} = 2 \Rightarrow 2k = \ln 2 \Rightarrow k = \frac{1}{2}\ln 2$$

$$60 = C\underbrace{e^{-3\ln 2}}_{e^{\ln(\frac{1}{2^3})}} \Rightarrow 60 = C \cdot \frac{1}{2^3} \Rightarrow C = 60 \cdot 8 = 480$$

$$\boxed{y(t) = 20 + 480e^{-\frac{1}{2}(\ln 2)t} = 20 + 480 \cdot \left(\frac{1}{2}\right)^{t/2}}$$

(3) $y(0) = 20 + 480 = 500°C$

$e^x + xy - \frac{1}{2}y^2 = \frac{1}{2}$

Exercise 3 **((1+1)+(5+5)=12 points)**

For each of the following inital value problems (a) and (b): $y^2 - 2xy - 2e^x + 1 = 0$

1. Identify the differential equation (linear, nonlinear, separable, exact), $y = x \pm \sqrt{x^2 + 2e^x - 1}$
2. Find the solution (you may leave your solution in an implicit form).

(a) $y' = \dfrac{e^x + y}{y - x}$ with initial condition $y(0) = 1$.

(b) $(x^2 + 1)y + xy' = x$ with initial condition $y(1) = 2$.

(a) Write the DE as $\underbrace{e^x + y}_{M(x,y)} + \underbrace{(x-y)}_{N(x,y)}y' = 0$. Then $\frac{\partial M}{\partial y} = 1 = \frac{\partial N}{\partial x}$. So the DE is exact.

It is non linear (because of the term yy') and not separable (because of the term $x-y$).

We look for $\Psi(x,y)$ such that $M = \frac{\partial \Psi}{\partial x}$ and $N = \frac{\partial \Psi}{\partial y}$. Hence

$\Psi(x,y) = \int (e^x + y)\,dx + h(y) = e^x + yx + h(y)$, and $\frac{\partial \Psi}{\partial y} = N(x,y) = x - y$, i.e.

$x - y = x + h'(y)$. Hence $h'(y) = -y$, i.e. $h(y) = -\frac{1}{2}y^2$ (up to a constant)

So $\Psi(x,y) = e^x + xy - \frac{1}{2}y^2$. The general solution is $e^x + xy - \frac{1}{2}y^2 = C$, C constant.

We fix the value of the constant using the initial condition $y(0) = 1$, which yields $e^0 - \frac{1}{2} = C$. Thus $e^x + xy - \frac{1}{2}y^2 = \frac{1}{2}$.

[To get the explicit solution, one can solve this quadratic equation for y and obtain $y = x \pm \sqrt{x^2 + 2e^x - 1}$. The sign "+" has to be chosen, since $y(0) = 1$.]

(b) This DE is linear, non separable and not exact [if we write it as

$\underbrace{[(x^2+1)y - x]}_{M(x,y)} + \underbrace{x}_{N(x,y)}y' = 0$, then $\frac{\partial M}{\partial y} = x^2 + 1 \neq 1 = \frac{\partial N}{\partial x}$]

In standard form: $y' + \frac{x^2+1}{x}y = 1$, with $x \neq 0$.

$\int \frac{x^2+1}{x}\,dx = \int x\,dx + \int \frac{1}{x}\,dx = \frac{1}{2}x^2 + \ln x + C$. $\leftarrow$ [The initial condition is at $x = 1$; so can suppose $x > 0$]

The integrating factor is therefore $\mu(x) = e^{\frac{1}{2}x^2 + \ln x} = x e^{\frac{1}{2}x^2}$

Thus $(\mu(x)y(x))' = x e^{\frac{1}{2}x^2}$, which gives $\mu(x)y(x) = \int x e^{\frac{1}{2}x^2}\,dx = e^{\frac{1}{2}x^2} + C$

i.e. $y(x) = \frac{1}{x} + \frac{C}{x} e^{-\frac{1}{2}x^2}$. Since $2 = y(1) = 1 + Ce^{-\frac{1}{2}}$, we have $C = e^{1/2}$.

Conclusion: $y(x) = \frac{1}{x}\left(1 + e^{\frac{1}{2}(1-x^2)}\right)$.

Exercise 4 (3+4+3+4+2=16 points)
A mass of 3 kg is attached to spring with spring constant 75 N/m. Suppose that there is no damping. The mass is initially displaced 0.2 m downward from its equilibrium position and given an upward velocity of 0.5 m/sec.

Suppose first that no external force acts on the system.

1. Determine the initial value problem describing the movement of the mass.
2. Find the position of the mass at any time t.
3. Show that the motion is periodic. Determine its period and its amplitude.

Suppose now that a periodic external force $F(t) = 10\cos(5t)$ N acts on the system.

4. Find the position of the mass at any time t under the same initial conditions as above.
5. Describe the motion of the mass for large values of t.

1. $my''+ky=0$, i.e. $3y''+75y=0$, i.e. $\boxed{y''+25y=0 \ , \ y(0)=0.2, \ y'(0)=-0.5}$

2. Characteristic equation: $\lambda^2+25=0$, $\lambda=\pm 5i$ purely imaginary complex conjugate eigenvalues
The general solution of $y''+25y=0$ is
$y(t)=C_1\cos(5t)+C_2\sin(5t)$, C_1, C_2 constants fixed by the initial conditions.
Notice that : $y'(t)=-5C_1\sin(5t)+5C_2\cos(5t)$. So $\begin{cases} 0.2 = C_1 \\ -0.5 = 5C_2 \end{cases} \Rightarrow \begin{cases} C_1 = \frac{1}{5} \\ C_2 = -\frac{1}{10} \end{cases}$.
The position of the mass at time t is $\boxed{y(t)=\frac{1}{5}\cos(5t)-\frac{1}{10}\sin(5t)}$

3. $y(t)$ is a periodic function of period $\frac{2\pi}{5}$ because linear combination of $\cos(5t)$ and $\sin(5t)$. The amplitude is $R=\sqrt{\frac{1}{25}+\frac{1}{100}}=\sqrt{\frac{1}{20}}=\frac{1}{2\sqrt{5}}$

4. The IVP describing the system is now $3y''+75y=10\cos(5t)$, with the same initial conditions $y(0)=0.2$ and $y'(0)=-0.5$. The general solution is of the form $y(t)=y_c(t)+Y(t)$ where $y_c(t)=C_1\cos(5t)+C_2\sin(5t)$ as in 2. and $Y(t)$ a particular solution.
Apply the method of undetermined coefficients. Since $\cos(5t)$ is a solution of the associated homog. DE, we set
$Y(t)=t(A\cos(5t)+B\sin(5t))$. Then $\begin{cases} Y'(t)=A\cos(5t)+B\sin(5t)+5t(-A\sin(5t)+B\cos(5t)) \\ Y''(t)=2(-5A\sin(5t)+5B\cos(5t))+25t(-A\cos(5t)-B\sin(5t)) \end{cases}$
Inserting into the DE:
$3Y''(t)+75Y(t)=10\cos(5t) \Leftrightarrow -30A\sin(5t)+30B\cos(5t)-75\cancel{At}\cos(5t)-75B\cancel{t}\sin(5t)+$
$75\cancel{At}\cos(5t)+75\cancel{B}t\sin(5t)=10\cos(5t)$
So $A=0, B=1/3$, i.e. $Y(t)=\frac{1}{3}t\sin(5t)$

$\left.\begin{array}{l} y(t)=C_1\cos(5t)+C_2\sin(5t)+\frac{1}{3}t\sin(5t) \\ y'(t)=-5C_1\sin(5t)+5C_2\cos(5t)+\frac{1}{3}\sin(5t)+\frac{5}{3}t\cos(5t) \end{array}\right\} \Rightarrow \begin{cases} C_1=\frac{1}{5} \\ C_2=-\frac{1}{10} \end{cases}$ as before

Thus $\boxed{y(t)=\frac{1}{5}\cos(5t)-\frac{1}{10}\sin(5t)+\frac{1}{3}t\sin(5t)}$

5. Periodic oscillations with semi-period $\frac{2\pi}{5}$, but with increasing amplitude
$\frac{1}{3}\left(\frac{\pi}{2}+2\pi n\right)\cdot\frac{1}{5}=\frac{\pi}{30}+\frac{2\pi n}{15} \to +\infty$ as $n\to+\infty$.

Exercise 5 **((3+3)+6=12 points)**

1. For each the following initial value problems, determine the largest possible interval on which the solution exists and is unique:

$$t(t-4)\,\frac{dy}{dt} - 2ty = (t-4)^2, \qquad y(5) = 1\,. \tag{1}$$

$$t(t-4)\,\frac{dy}{dt} - 2ty = \frac{1}{\sin t}, \qquad y(5) = 1\,, \tag{2}$$

Justify your answer. *(Do not attempt to solve the differential equations.)*

2. Using the method of the integrating factor, solve the initial value problem (1).

1. Write the DE in the form $\frac{dy}{dt} - \frac{2}{t-4}y = \frac{t-4}{t}$ (1)

$\frac{dy}{dt} - \frac{2}{t-4}y = \frac{1}{t(t-4)\sin t}$ (2)

(1) and (2) are linear DE, now in standard form $y'(t) + p(t)y(t) = g(t)$ where $p(t) = -\frac{2}{t-4}$ and $g(t) = \frac{t-4}{t}$ for (1) and $g(t) = \frac{1}{t(t-4)\sin t}$

The solution of the DE exists and is unique in the largest open interval I containing $t_0 = 5$ on which both $p(t), g(t)$ are continuous.

(1): $p(t)$ is continuous on $(-\infty,4)\cup(4,+\infty)$
$g(t)$ — " — on $(-\infty,0)\cup(0,+\infty)$ $\Big\} \Rightarrow I = (4,+\infty)$

(2) $p(t)$ is continuous on $(-\infty,4)\cup(4,+\infty)$
$g(t)$ is continuous on $(-\infty,0)\cup(0,4)\cup(4,+\infty) \setminus \{k\pi;\ k\in\mathbb{Z}\}$ $\Big\} \Rightarrow I = (4, 2\pi)$

2. The integrating factor is $\mu(t) = e^{-2\int \frac{1}{t-4}dt} = e^{-2\ln(t-4)} = e^{\ln\frac{1}{(t-4)^2}} = \frac{1}{(t-4)^2}$

$(\mu(t)y(t))' = \frac{(t-4)}{t(t-4)^2} = \frac{1}{t(t-4)} = \frac{A}{t} + \frac{B}{t-4} = -\frac{1}{4}\frac{1}{t} + \frac{1}{4}\frac{1}{t-4}$

i.e. $\mu(t)y(t) = -\frac{1}{4}\int \frac{1}{t}dt + \frac{1}{4}\int \frac{1}{t-4}dt = -\frac{1}{4}\ln|t| + \frac{1}{4}\ln|t-4| + C = \frac{1}{4}\ln\left|\frac{t-4}{t}\right| + C$

Hence $y(t) = (t-4)^2\left[\frac{1}{4}\ln\left|\frac{t-4}{t}\right| + C\right]$, C constant

$1 = y(5) = \frac{1}{4}\ln\left(\frac{1}{5}\right) + C \Rightarrow C = 1 + \frac{1}{4}\ln 5$. Thus $y(t) = (t-4)^2\left[\frac{1}{4}\ln\left|\frac{t-4}{t}\right| + 1 + \frac{1}{4}\ln 5\right]$

$= (t-4)^2\left[\frac{1}{4}\ln\left(\frac{t-4}{t}\right) + 1 + \frac{1}{4}\ln 5\right]$ because this solution is for $t\in(4,+\infty)$, so $\frac{t-4}{t} > 0$

Exercise 6 (2+10=12 points)

Consider the piecewise defined function $g(t) = \begin{cases} 2t & \text{if } 0 \le t < 1 \\ 0 & \text{if } t \ge 1 \end{cases}$

1. Express $g(t)$ in terms of unit step functions.
2. Find the solution of the initial value problem

$$y'' + y' = g(t) \quad \text{with} \quad y(0) = 0 \text{ and } y'(0) = 0$$

1. $g(t) = 2t(u_0 - u_1) = 2t - 2t\,u_1(t)$

2. Notice first that $\mathcal{L}\{g\}(s) = 2\mathcal{L}\{t\}(s) - 2\mathcal{L}\{t\,u_1(t)\}(s)$

$$= \frac{2}{s^2} - 2\mathcal{L}\{t+1\}(s)e^{-s}$$

$$= \frac{2}{s^2} - 2\left(\frac{1}{s^2} + \frac{1}{s}\right)e^{-s} \quad \text{for } s > 0$$

$\mathcal{L}\{h(t-1)u_1(t)\} = \mathcal{L}\{h\}(s)e^{-s}$

$h(t-1) = t \Rightarrow h = t+1$

Apply the Laplace transform to both sides of the DE:

$\mathcal{L}\{y''\}(s) + \mathcal{L}\{y'\}(s) = \mathcal{L}\{g\}(s)$. Set $Y = \mathcal{L}\{y\}$

$(s^2 Y(s) - s\underbrace{y(0)}_{=0} - \underbrace{y'(0)}_{=0}) + (sY(s) - \underbrace{y(0)}_{=0}) = \mathcal{L}\{g\}(s)$, i.e. $s(s+1)Y(s) = \mathcal{L}\{g\}(s)$. Thus

$$s(s+1)Y(s) = \frac{2}{s^2} - 2\left(\frac{1}{s^2} + \frac{1}{s}\right)e^{-s} = \frac{2}{s^2} - 2\,\frac{s+1}{s^2}e^{-s}, \text{ i.e. } Y(s) = \frac{2}{s^3(s+1)} - \frac{2}{s^3}e^{-s}$$

Partial fraction decomposition: $\frac{1}{s^3(s+1)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s^3} + \frac{D}{s+1} = \frac{1}{s} - \frac{1}{s^2} + \frac{1}{s^3} - \frac{1}{s+1}$

By linearity of $\mathcal{L}^{-1}$:

$$y = \mathcal{L}^{-1}\{Y\} = 2\left[\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s^3}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\}\right] - 2\mathcal{L}^{-1}\left\{\frac{1}{s^3}e^{-s}\right\}$$

i.e. $y(t) = 2\left(1 - t + \frac{1}{2}t^2 - e^{-t}\right) - 2h(t-1)u_1(t)$ where $\mathcal{L}\{h\}(s) = \frac{1}{s^3}$, i.e. $h(t) = \frac{1}{2}t^2$

$$= 2\left(1 - t + \frac{1}{2}t^2 - e^{-t}\right) - (t-1)^2 u_1(t)$$

Exercise 7 (6+6=12 points)

~~**1.** Find the Laplace transform of~~

$$\cancel{f(t) = 2e^{t+1}(1 - \delta(t)) + \int_0^t (t - \tau)^2 \sin(3\tau)\, d\tau}$$

2. Determine the inverse Laplace transform of the function

$$F(s) = \frac{e^{-2s}}{(s+1)^2} + \frac{1}{s^2 + 2s + 2}\,.$$

2. $\mathcal{L}\{h\}(s)e^{-2s} = \mathcal{L}\{h(t-2)\,u_2(t)\}(s$

$\mathcal{L}\{h\}(s) = \dfrac{1}{(s+1)^2} = \dfrac{1}{s^2}\Big|_{s \to s+1} \Rightarrow h(t) = te^{-t}$

$h(t-2) = (t-2)\, e^{-(t-2)}$

$\dfrac{1}{s^2+2s+2} = \dfrac{1}{(s+1)^2+1} = \mathcal{L}\{e^{-t}\sin(t)\}(s)$

Thus $\mathcal{L}^{-1}\{F\}(t) = \mathcal{L}^{-1}\left\{\dfrac{e^{-2s}}{(s+1)^2}\right\}(t) + \mathcal{L}^{-1}\left\{\dfrac{1}{s^2+2s+2}\right\}(t)$

$= (t-2)\,e^{-(t-2)}\,u_2(t) + e^{-t}\sin(t)$

Exercise 8 (2+2+12=16 points)
Consider the following system of differential equations:

$$\frac{dx}{dt} = x(2+x-y)$$
$$\frac{dy}{dt} = y(1+x)$$

1. Find all the critical points of the system.
2. Compute the Jacobian matrix for the system.
3. For each critical point, find the corresponding approximating linear system. Find the eigenvalues of each linear system and classify each critical point according to type (nodal, spiral, center,...) and stability (asymptotically stable, stable, or unstable).

1. Critical points are the solutions of $\begin{cases} dx/dt = 0 \\ dy/dt = 0 \end{cases}$, i.e. $\begin{cases} F(x,y) = 0 \\ G(x,y) = 0 \end{cases}$ where

$F(x,y) = x(2+x-y)$
$G(x,y) = y(1+x)$. $\begin{cases} x(2+x-y) = 0 \\ y(1+x) = 0 \end{cases} < \begin{matrix} y = 0 \\ x = -1 \end{matrix}$ i.e. $(0,0), (-2,0), (-1,1)$ are the critical points

2. $J(x,y) = \begin{pmatrix} F_x & F_y \\ G_x & G_y \end{pmatrix} = \begin{pmatrix} 2+2x-y & -x \\ y & 1+x \end{pmatrix}$

3. $(0,0)$: approximating linear system $\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$

$\begin{vmatrix} 2-\lambda & 0 \\ 0 & 1-\lambda \end{vmatrix} = (\lambda-2)(\lambda-1) = 0 \; . \Rightarrow \lambda_1 = 1, \lambda_2 = 2$

Two positive real distinct eigenvalues. So, the critical point $(0,0)$ is is a node and is unstable

$(-2,0)$ $\frac{d}{dt}\begin{pmatrix} u \\ w \end{pmatrix} = \begin{pmatrix} -2 & 2 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} u \\ w \end{pmatrix}$ where $\begin{cases} u = x+2 \\ w = y \end{cases}$

$\begin{vmatrix} -2-\lambda & 2 \\ 0 & -1-\lambda \end{vmatrix} = (\lambda+2)(\lambda+1) = 0 \Leftrightarrow \lambda_1 = -1, \lambda_2 = -2$. Two negative distinct eigenvalues

So $(-2,0)$ is a node and is asymptotically stable.

$(-1,1)$ $\frac{d}{dt}\begin{pmatrix} u \\ w \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} u \\ w \end{pmatrix}$ where $\begin{cases} u = x+1 \\ w = y-1 \end{cases}$

$\begin{vmatrix} -1-\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = \lambda(\lambda+1) - 1 = \lambda^2 + \lambda - 1$

$\lambda^2 + \lambda - 1 \;\; 0 \Leftrightarrow \lambda = \frac{-1 \pm \sqrt{1+4}}{2}$ i.e. $\lambda_1 = \frac{-1+\sqrt{5}}{2} > 0 > \lambda_2 = \frac{-1-\sqrt{5}}{2}$

$(-1,1)$ is a saddle point and is unstable.