Georgia Tech – Lorraine
Fall 2019
Differential Equations
Math 2552
5/12/2019

Last Name:
First Name:

Final Exam

Total time: 2 hours and 50 minutes.
Total points: 100 points

Please organize your work clearly, neatly, and legibly.
Identify your answers.
Show your work and justify your answers.
If you need extra space, use the back sides of each page.
Please do not use red or pink ink.

Calculators, notes, cell phones, and books are not allowed.
A table of Laplace transforms is provided.

I wish you success in your exam.

EX	points	
1	10	
2	10	
3	12	
4	16	
5	12	
6	12	
7	12	
8	16	
TOT	100	

Exercise 1 (2+3+5=10 points)
Consider the differential equation $y' = f(y)$ where $f(y) = -y(y-1)^2$.

1. Sketch the graph of $f(y)$ versus y.
2. Determine the critical (or equilibrium) points.
3. Classify each critical point as asymptotically stable, unstable or semistable. Draw the phase line and sketch several graphs of solutions in the ty-plane.

Exercise 2 (3+5+2=10 points)
A hot metal bar is placed in a room at a constant temperature of 20° C. After 6 minutes the temperature of the bar is measured as 80° C. Two minutes later, the temperature of the bar has decreasead to 50° C.
Suppose that Newton's law of cooling applies with transmission coefficient k.

1. Write an initial value problem modeling the temperature of the bar as a function of time.
2. Solve the initial value problem. The transmission coefficient k has to be computed. (Leave your answer in term of ln.)
3. What was the initial temperature of the metal bar?

Exercise 3 ((1+1)+(5+5)=12 points)
For each of the following inital value problems (a) and (b):

1. Identify the differential equation (linear, nonlinear, separable, exact),
2. Find the solution (you may leave your solution in an implicit form).

(a) $y' = \dfrac{e^x + y}{y - x}$ with initial condition $y(0) = 1$.

(b) $(x^2 + 1)y + xy' = x$ with initial condition $y(1) = 2$.

Exercise 4 (3+4+3+4+2=16 points)
A mass of 3 kg is attached to spring with spring constant 75 N/m. Suppose that there is no damping. The mass is initially displaced 0.2 m downward from its equilibrium position and given an upward velocity of 0.5 m/sec.

Suppose first that no external force acts on the system.

1. Determine the initial value problem describing the movement of the mass.
2. Find the position of the mass at any time t.
3. Show that the motion is periodic. Determine its period and its amplitude.

Suppose now that a periodic external force $F(t) = 10\cos(5t)$ N acts on the system.

4. Find the position of the mass at any time t under the same initial conditions as above.
5. Describe the motion of the mass for large values of t.

Exercise 5 ((3+3)+6=12 points)

1. For each the following initial value problems, determine the largest possible interval on which the solution exists and is unique:

$$t(t-4)\,\frac{dy}{dt} - 2ty = (t-4)^2, \qquad y(5) = 1\,. \tag{1}$$

$$t(t-4)\,\frac{dy}{dt} - 2ty = \frac{1}{\sin t}, \qquad y(5) = 1\,, \tag{2}$$

   Justify your answer. *(Do not attempt to solve the differential equations.)*

2. Using the method of the integrating factor, solve the initial value problem (1).

Exercise 6 (2+10=12 points)

Consider the piecewise defined function $g(t) = \begin{cases} 2t & \text{if } 0 \leq t < 1 \\ 0 & \text{if } t \geq 1 \end{cases}$

1. Express $g(t)$ in terms of unit step functions.

2. Find the solution of the initial value problem

$$y'' + y' = g(t) \quad \text{with} \quad y(0) = 0 \;\text{ and }\; y'(0) = 0$$

Exercise 7 (6+6=12 points)

1. Find the Laplace transform of

$$f(t) = 2e^{t+1}\big(1 - \delta(t)\big) + \int_0^t (t-\tau)^2 \sin(3\tau)\, d\tau$$

2. Determine the inverse Laplace transform of the function

$$F(s) = \frac{e^{-2s}}{(s+1)^2} + \frac{1}{s^2+2s+2}\,.$$

Exercise 8 (2+2+12=16 points)
Consider the following system of differential equations:

$$\begin{aligned}\frac{dx}{dt} &= x(2+x-y)\\ \frac{dy}{dt} &= y(1+x)\end{aligned}$$

1. Find all the critical points of the system.
2. Compute the Jacobian matrix for the system.
3. For each critical point, find the corresponding approximating linear system. Find the eigenvalues of each linear system and classify each critical point according to type (nodal, spiral, center,...) and stability (asymptotically stable, stable, or unstable).