Final Exam Review

Chapter 1: Introduction

Section 1.1: Terminology: (ordinary) differerential equation, independent and dependent variables, solutions, integral curves, general solution. Use DE's for modeling: Newton's law of cooling, population growth models (see also Section 2.5). See Section 2.3 for mixing problems.

Section 1.2 and Section 2.5: Autonomous differential equations: find equilibrium solutions (also called critical points or stationary points), draw phase lines, sketch integral curves. Determine if a critical point is asymptotically stable, semistable or unstable. Drawing of direction fields is not requested.

Section 1.3: classification of DE's: order, linear/non linear, homogeneity. Initial value problems (IVP).

Chapter 2: First order differential equations

Section 2.1: solve separable equations.

Section 2.2: standard form of a first order linear DE. Solve first order linear differential equations by using integrating factors.

Section 2.3: modeling: write down a differential equation to model a problem and then solve the differential equation (or the IVP). Mixing problems.

Section 2.4: existence and uniqueness of solutions: first order linear DE (Theorem 2.4.1) and first order non-linear DE (Theorem 2.4.2).

Section 2.5: see Section 1.2.

Section 2.6: recognize a first order exact DE and solve it.

Section 2.7 is not on the final.

Chapter 3: Systems of two first order equations

Section 3.1: Systems of two linear equations. Homogenous systems. Matrix notation. Matrix of coefficients of the system. Trace and determinant of a 2×2 matrix. Invertible matrices. A matrix is invertible if and only if its determinant is non-zero. Inverse of a matrix. Solutions of linear systems. Characteristic polynomial and characteristic equation. Eigenvalues and eigenvectors.

Section 3.2: Systems of two first-order linear DE's. Solutions. Initial value problems (IVP). Theorem on the existence and uniqueness of the solutions of an IVP for a system of two linear DE's (Theorem 3.2.1). Matrix notation, vector solution. Special cases: homogenous systems; systems with constant coefficients. Transform a second order linear DE into a system of first order linear DE's.

Section 3.3: Homogeneous systems $\mathbf{x}' = \mathbf{A}\mathbf{x}$ with constant coefficients.
The superposition principle (Theorem 3.3.1), Wronskian and linear independence, notion of fundamental system of solutions, general solution (Theorem 3.3.4).
Constructing solutions using eigenvalues and eigenvectors of $\mathbf{A}$.
The general solution when $\mathbf{A}$ admits two linearly independent eigenvectors.

For a homogeneous system $\mathbf{x}' = \mathbf{A}\mathbf{x}$: Superposition principle (Theorem 3.3.1), Wronskian and linear independence, fundamental sets of solutions, general solution (Theorem 3.3.4).

Sections 3.3, 3.4 and 3.5 (analytic methods): Solve the homogeneous system $\mathbf{x}' = \mathbf{A}\mathbf{x}$:

- Find a fundamental set of solutions and write the general solution (depending on the eigenvalues of $\mathbf{A}$).
- When $\mathbf{A}$ has complex (conjugate) eigenvalues, write the solution in terms of real solutions (Section 3.4).

• Behavior of the solutions when $\mathbf{A}$ has two distinct real eigenvalues (Table 3.3.1) or complex eigenvalues (Table 3.4.1) or repeteated eigenvalues (Table 3.5.1).

Sections 3.2, 3.3, 3.4 and 3.5 (geometric methods): Component plots of solutions. Autonomous systems: notions of phase plane, trajectories, direction fields, equilibrium point (or equilibrium point or critical point), phase portrait.

For a homogeneous system $\mathbf{x}' = \mathbf{A}\mathbf{x}$:

• Determine if $(0, 0)$ is a nodal sink/source, saddle, spiral sink/source, or a center. Stability.

• Behavior of the solutions when $\mathbf{A}$ has two distinct real eigenvalues (Table 3.3.1) or complex eigenvalues (Table 3.4.1) or repeteated eigenvalues (Table 3.5.1).

• Sketch some trajectories in the phase plane.

Sections 3.6: nonlinear systems of two differential equations. Vector notation. Existence and uniqueness of the solutions (Theorem 3.6.1). Autonomous systems (see also Chapter 7).

Chapter 6: Systems of first-order linear equations

Section 6.1: Systems of n linear first-order DE's; matrix notation. The system of first order linear DE's associated with a linear n-th order differential equation; correspondence of initial value problems in this context.

Section 6.2: Existence and unicity of solutions for systems of first-order linear equations.
For a homogeneous system $\mathbf{x}' = \mathbf{A}\mathbf{x}$: Superposition principle (Theorem 6.2.2), Wronskian and linear independence, fundamental sets of solutions, general solution (Theorem 6.2.6).

Section 6.3: Defective and nondefective matrices. The general solution of a linear system $\mathbf{x}' = \mathbf{A}\mathbf{x}$ where $\mathbf{A}$ is nondefective with real eigenvalues (Theorem 6.3.1)

Section 6.4: Real-valued general solutions of $\mathbf{x}' = \mathbf{A}\mathbf{x}$ where $\mathbf{A}$ is nondefective and not all eigenvalues of $\mathbf{A}$ are real.

Chapter 4: Second order linear equations

Section 4.1: Second order linear equations: standard form, homogeneous/nonhomogenous, constant coefficients/variable coefficients, initial value problems (IVP).
See below for spring-mass system models.

Section 4.2: The system of first order linear differential equations associated with a second order linear differential equation, correspondence of initial conditions, matrix notation.
Existence and uniqueness of the solutions of an IVP for a 2nd order linear DE (Theorem 4.2.1).
Second order linear homogenous DE's: principle of superposition for a 2nd order DE (Theorem 4.2.2, Corollary 4.2.3) and for a homogenous system of 1st order linear DE's (Theorem 4.2.4, Corollary 4.2.5).
Wronskian of two solutions, fundamental solutions and general solution: for homogenous systems of two 1st order linear DEs (Theorem 4.2.6) and for 2nd order linear DE (Theorem 4.2.7).

Section 4.3: Second-order linear homogenous DE's with constant coefficients: characteristic equation, fundamental system of solutions constructed from the roots of the characteristic equation (Theorem 4.3.1, for the DE and for its associated system), general solution (Theorem 4.3.2).

Section 4.5: Solution of a second-order linear nonhomogenous DE: the general solution as a sum of the general solution of the corresponding homogenous DE (complementary solution) and one particular solution (Theorems 4.5.1 and 4.5.2).

Sections 4.7: The method of variation of parameters is not on the program of the final. You are welcome to use it instead of the method of undetermined coefficients unless otherwise requested.

Sections 4.1 and 4.4 : Spring-mass systems

- *Section 4.1:* the model: mass, spring constant, damping factor.
- *Section 4.4:* unforced or free systems (harmonic oscillators).

Undamped free system: phase-amplitude form of the general solution (period, natural frequency, phase, amplitude).

Damped free system: underdamped, critically damped or overdamped harmonic motion; critical damping; quasi-frequency and quasi-period of an underdamped harmonic motion.

Chapter 5: The Laplace transform

Section 5.1: Improper integrals: examples and tests of convergence (Theorem 5.1.4). Piecewise continuous functions. Functions of exponential order. The Laplace transform $\mathcal{L}$: definition, linearity (Theorem 5.1.2), Laplace transform of piecewise continuous functions of exponential order (Theorem 5.1.6, Corollary 5.1.7).

Section 5.2: Laplace transforms of $e^{ct}f$, of $t^n f(t)$, of derivatives, of differential equations.

Section 5.3: Inverse Laplace transform $\mathcal{L}^{-1}$, linearity, partial fraction decompositions.

Sections 5.4 and 5.6: Solving differential equations with Laplace transforms.

Section 5.5: unit step functions, indicator functions and their Laplace transforms. Representations of piecewise defined functions. Laplace transforms of time-shifted functions.

The Laplace transforms of periodic functions are not on the final exam.

Chapter 7: Non-linear differential equations and stability

Section 7.1: autonomous systems of two first order DE's. Critical points and stability (precise definitions). The oscillating pendulum.

Section 7.2: Isolated critical points, almost linear system, Jacobian matrix, linear approximation of an almost linear system. Stability and instability properties of almost linear systems in relation to those of their linear approximations (Table 7.2.2). The damped pendulum.

Sections 7.3 and 7.4: Examples of almost linear systems: competing species models and prey-predator equations.

Chapter 8 is not on the final exam.