Section 1.3: Classification of Differential Equations

Classification allows us to determine which methods we can use to solve a DE.

- Number of independent variables:
  - ◇ Ordinary Differential Equations (ODE)
  - ◇ Partial Differential Equations (PDE)
- Order of an ODE
- Linearity:
  - ◇ Linear Differential Equations
  - ◇ Homogeneous Linear Differential Equations
  - ◇ Standard form

Number of independent variables

- **Ordinary Differential Equation (ODE)**: the functions in the DE depend on a single independent variable.
- **Partial Differential Equation (PDE)**: the functions in the DE depend on more than one independent variable.

Remark: The DE as defined at the beginning of the course are in fact ODE. This course focuses on ODEs only. When talking about a DE, we shall always mean an ODE.

Examples

(1) Newton's Law of Cooling

$$\frac{du}{dt} = -k(u - T)$$

is an example of an ODE. The unique independent variable is t.

(2) The heat equation

$$\frac{\partial u(x,t)}{\partial t} = D\frac{\partial^2 u(x,t)}{\partial x^2}$$

is an example of a PDE.

Model of the temperature $u(x,t)$ of a metal rod. Two independent variables: t=time, x position along the rod

The Order of an ODE

Definition

The **order** of an ODE is the highest degree derivative which appears in the equation.

Examples

(1) Newton's Law of Cooling $u' = -k(u - T)$ has order 1.

(2) The ODE $y''' + 2e^t y'' + yy' = t^4$ has order 3.

(3) $y'' + (y')^{10} = 4$ has order 2.

Remark: This course mostly focuses on ODE's of order 1 and 2.

Linearity / Homogeneity

Definition

An n**-th order linear ODE** is an ODE of the form:

$$a_0(t)y^{(n)}(t) + a_1(t)y^{(n-1)}(t) + \cdots + a_n(t)y(t) = g(t)$$

where

- $a_0(t)$, $a_1(t)$, ..., $a_n(t)$ and $g(t)$ are functions of t which are given (called the **coefficients**)
- $y(t)$ is the unknown function.
- $y^{(k)}(t)$ denotes the k-th derivative of y. **E.g.** $y^{(1)} = y'$, $y^{(2)} = y''$...

If $g(t) = 0$ for all t, we say that the linear ODE is **homogeneous**.

An ODE which is not of the above form is called **non-linear**.

Examples

(1) Newton's Law of Cooling $\frac{d}{dt}u(t) = -k(u - T)$ is linear, has order 1, is nonhomogeneous if $T \neq 0$.

(2) $t^3y'' + cos(t)y' = y$ is linear, of order 2, homogeneous.

(3) $yy'' + y' = 0$ is non-linear, of order 2

(4) $y + y' = \sin(y + t)$ is non-linear, of order 1.

First order linear ODEs

A first order linear ODE is of the form

$$a_0(t)\frac{dy}{dt} + a_1(t)y = g(t)$$

If $a_0(t) = 0$ for all t, there is no DE (no derivative)!
If not, for all t so that $a_0(t) \neq 0$, we can divide both sides of the DE by $a_0(t)$:

$$\frac{dy}{dt} + \frac{a_1(t)}{a_0(t)}y = \frac{g(t)}{a_0(t)}$$

Set

$$p(t) = \frac{a_1(t)}{a_0(t)} \quad \text{and} \quad h(t) = \frac{g(t)}{a_0(t)}$$

Then the above equation can be put in the **standard form** (or normal form)

$$\frac{dy}{dt} + p(t)y = h(t)$$

Example
$t^3y' + ty = y + t$ is linear, of order 1, nonhomogeneous.
For all $t \neq 0$:

$$y' + \frac{t-1}{t^3}y = \frac{1}{t^2} \qquad \text{(standard form)}$$