Chapter 2: First order ODEs

$$\frac{dy}{dx} = f(x, y)$$

where

- x is the independent variable
- y is the unknown function
- f is a function of two variables

Two main classes of first order ODE for which where are general methods to find solutions:

- separable equations
- linear differential equations

A solution is a differentiable function $y = \phi(x)$ which satisfies the DE *for all* x *is some interval* (we will be more precise on this).

Section 2.1: Separable equations

Definition

Consider the first order DE

$$\frac{dy}{dx} = f(x, y)$$

where y is the unknown function, x is the independent variable.

The DE is called **separable** if the function f can be split as a product

$$f(x, y) = p(x)q(y)$$

where

- p is a function depending on x only,
- q is a function depending on y only.

Examples

- a, b constants. Then $\frac{dy}{dx} = ay + b$ is separable [Here $p(x) = 1$, $q(y) = ay + b$]

  e.g. Newton's Law of Cooling with constant ambient temperature T_0: $u' = -k(u - T_0)$
- $y' = -xy^2$ of order 1, non-linear, separable
- $\frac{dy}{dx} = y + x$ of order 1, linear, not separable.

General method to solve a separable DE: $\dfrac{dy}{dx} = p(x)q(y)$

- If $q(y) = 0$ then $f(x, y) = 0$ and the solutions are constant functions.
- So suppose $q(y) \neq 0$ (for certain values of y). Divide both sides of the DE by $q(y)$ and get

$$\frac{1}{q(y)}\,\frac{dy}{dx} = p(x)$$

- Integrate both sides with respect to x:

$$\int \frac{1}{q(y)}\,\frac{dy}{dx}\,dx = \int p(x)\,dx$$

- Use the change of variables formula
  (equivalently, the relation $dy = \frac{dy}{dx}\,dx$ for the differential dy of y in terms of the differential dx)
  and get:

$$\int \frac{1}{q(y)}\,dy = \int p(x)dx$$

Let $Q(y)$ and $P(x)$ be antiderivatives of $\frac{1}{q(y)}$ and $p(x)$, respectively, i.e.

$$Q'(y) = \frac{1}{q(y)} \quad \text{and} \quad P'(x) = p(x)$$

Then

$$\int \frac{1}{q(y)}\, dy = \int p(x) dx$$

is equivalent to

$$Q(y) = P(x) + C, \qquad C\text{=constant}$$

These are **implicit solutions**.

To find **explicit solutions** $y = \phi(x)$ one needs to solve $Q(y) = P(x) + C$ for y.

(this might be very difficult or even not possible).

Separable equations: examples

- Newton's law of cooling with constant ambient temperature T_0: see Chapter 1.
- Solve $y' = -xy^2$
- Determine the integral curves of the DE: $yy' = -x$.
  Solve the initial value problem $yy' = -x$, $y(0) = 1$ and determine the interval in which the solution exists.