Section 2.1: Separable equations

EXAMPLES

(1) Solve $y' = -xy^2$

This is a 1st order, non-linear, separable DE (of the form $y' = p(x)q(y)$ with $p(x) = -x$ and $q(y) = y^2$).

Since $q(y)=0$ has solution $y=0$, this is a constant solution.
So assume that $y \neq 0$ and divide both sides of the equation by $q(y) = y^2$:

$$\frac{1}{y^2}\frac{dy}{dx} = -x$$

Integrate wrt x: $\int \frac{1}{y^2}\underbrace{\frac{dy}{dx}dx}_{dy} = -\int x\,dx$ (*)

Since $\int \frac{1}{y^2}dy = -\frac{1}{y} + \text{constant}$ and $\int x\,dx = \frac{1}{2}x^2 + \text{constant}$

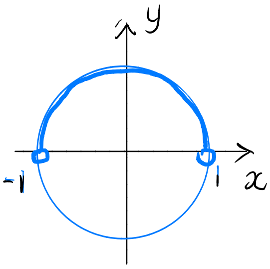
(*) becomes: $-\frac{1}{y} = -\frac{1}{2}x^2 + C_1$, $C_1 = \text{constant}$ i.e. $\frac{1}{y} = \frac{x^2 - 2C_1}{2}$

The general solution is $y = \frac{2}{x^2+C}$, $C = \text{constant}$

(2) (a) Determine the integral curves of the DE: $yy' = -x$.

(b) Solve the IVP $\begin{cases} yy' = -x \\ y(0) = 1 \end{cases}$ and determine the interval in which the solution exists

$yy' = -x$ is a 1st order, non-linear, separable DE (the variables are already separated). Integrating wrt to x we get



$$\int y \underbrace{\frac{dy}{dx}dx}_{dy} = -\int x\,dx$$

i.e. $y^2 + x^2 = C$, $C = \text{constant}\ (>0)$ [implicit solution]

The integral curves are circles of center O and radius $\sqrt{C}$

For each $C>0$, we have two types of solutions: $y = \sqrt{C - x^2}$ and $y = -\sqrt{C-x^2}$ defined on $-\sqrt{C} < x < \sqrt{C}$. Since $y(0) = 1\ (>0)$, the solution to the IVP is $y = \sqrt{1-x^2}$, defined on $-1 < x < 1$

Section 2.2: Linear DE: method of integrating factors

EXAMPLE $y' + 2ty = 2e^{-(t-1)^2}$

(1) Solve the DE using the method of integrating factors

(2) Determine the solution satisfying the initial condition $y(0)=1$

(3) Use the general solution to determine how solutions behave for $t \to +\infty$.

(1) The DE is in standard form $y' + p(t)y = h(t)$ with $p(t) = 2t$ and $h(t) = e^{-(t-1)^2}$

- Compute the integrating factor: $\mu(t) = e^{\int p(t)dt}$
  $\int p(t)dt = t^2 + C$. Can choose $\mu(t) = e^{t^2}$
- $\underbrace{\mu(t)y' + \mu(t)p(t)y}_{[\mu(t)y]'} = \mu(t)h(t)$ becomes

$$(e^{t^2}y)' = 2e^{t^2}e^{-(t-1)^2} = 2e^{t^2-(t^2-2t+1)} = 2e^{2t-1}$$

- Integrate $(e^{t^2}y)' = 2e^{2t-1}$ to get
  $e^{t^2}y = 2\int e^{2t-1}dt = e^{2t-1} + C$
- Solve for y: $y = e^{-t^2+2t-1} + Ce^{-t^2} = e^{-(t-1)^2} + Ce^{-t^2}$

(2) $1 = y(0) = e^{-1} + C \Rightarrow C = 1 - e^{-1}$, so $y = e^{-(t-1)^2} + (1-e^{-1})e^{-t^2}$ is the solution which satisfies $y(0)=1$

(3) $\lim_{t \to +\infty} e^{-(t-1)^2} + Ce^{-t^2} = e^{-(t-1)^2}$