First order linear ODE's: integrating factors

Consider the linear first order ODE in **standard form**

$$\frac{dy}{dt} + p(t)y = h(t)$$

Idea: multiply both sides of the ODE by a positive function $\mu(t)$

$$\mu(t)y' + \mu(t)p(t)y = \mu(t)h(t)$$

such that the left-hand side is the derivative of the product $\mu(t)y(t)$.

In other words, we want:

$$\mu'(t) = \mu(t)p(t)$$

to get:

$$(\mu(t)y)' = \mu(t)h(t)$$

We want: $\mu'(t) = \mu(t)p(t)$.

So we must have:

$$\frac{\mu'(t)}{\mu(t)} = p(t)$$

and therefore (by integrating both sides):

$$\mu(t) = e^{\int p(t)\, dt}$$

Remark: we can choose **one** antiderivative. The additive constant does not matter in the end (try it!)

μ is called the **integrating factor**.

The original DE reads now as follows:

$$(\mu(t)y)' = \mu(t)h(t)$$

Integrate it.

Conclusion: the solution of the DE: $\dfrac{dy}{dt} + p(t)y = h(t)$ is

$$y = \frac{1}{\mu(t)}\left(\int \mu(t)h(t)\, dt + constant\right)$$

Remark (disadvantage of this method): In general, $\displaystyle\int \mu(t)h(t)\, dt$ cannot be expressed in terms of elementary functions. One often uses numerical methods to approximate it.

The method of integrating factors

The procedure for solving a first order linear DE

$$\frac{dy}{dt} + p(t)y(t) = h(t)$$

1. Put the given DE in standard form $y' + p(t)y = h(t)$ (if necessary).
2. Calculate the integrating factor $\mu(t) = e^{\int p(t)\,dt}$
3. Multiply the DE by $\mu(t)$ and write it in the form $[\mu(t)y(t)]' = \mu(t)h(t)$.
4. Integrate
5. Solve for y

An example

Consider the first order linear DE

$$y' + 2ty = 2e^{-(t-1)^2}$$

(1) Solve the DE using the method of the integrating factor.
(2) Determine the solution with the initial condition $y(0) = 1$.
(3) Use the general solution to determine how solutions behave as $t \to +\infty$.