Section 2.4: Difference between linear and nonlinear ODE

In this section we are concerned with the two following questions:

Given the initial value problem (IVP):

$$\frac{dy}{dt} = f(t, y)$$
$$y(t_0) = y_0$$

1. **existence**: does the IVP have a solution, and if so, where?
2. **uniqueness**: is the solution unique?

We explore these questions for linear and non-linear cases.

The linear case

Theorem (Theorem 2.4.1)

Let $I = (\alpha; \beta)$ be an open interval and let $t_0 \in I$.
Consider the 1st order linear ODE in standard form:

$$y' + p(t)y = g(t)$$

Suppose that both $p(t)$ and $g(t)$ are continuous functions of $t \in I$.
Then there exists a unique function $y = \phi(t)$ satisfying the DE on I with the initial value condition

$$y(t_0) = y_0$$

(where y_0 is an arbitrarily fixed initial value).

Idea of the proof. The method of integrating factor provides the explicit solution. See textbook.

Example: $(t^2 - 4)y' + ty = e^t$ with $y(1) = 1$
Determine (without solving the DE) an interval in which the solution of the IVP exists.

The non-linear case

Theorem (Theorem 2.4.2)

Let $R = \{(t, y) : \alpha < t < \beta,\ \gamma < y < \delta\}$ be an open rectangle in the ty-plane and let $(t_0, y_0) \in R$. Consider the first order DE : $y' = f(t, y)$

Suppose that both f and $\dfrac{\partial f}{\partial y}$ are continuous on R.

Then, in some interval $I_0 = (t_0 - h, t_0 + h)$ contained in (α, β), there exists a unique function $y = \phi(t)$ satisfying the DE on I with the initial condition $y(t_0) = y_0$.

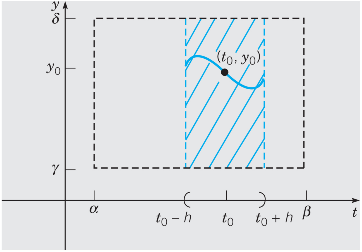

Illustration of *R* in Theorem 2.4.2 (Brennan & Boyce, Differential Equations, p. 72)

Example:
Does $y' = y^{1/3},\ y(0) = 0$
satisfy the conditions of this theorem?

Linear vs non-linear case

Remark: Take $f(t, y) = -p(t)y + g(t)$ so that $\frac{\partial f}{\partial y} = -p(t)$. Then $p(t), g(t)$ are continuous in $t \in I$ if and only if $f, \frac{\partial f}{\partial y}$ are continuous in $(t, y) \in R = I \times \mathbb{R}$.

- The 1st order ODE $y' + p(t)y = g(t)$ has the following properties.
  - If the p and g are continuous, there is a general solution (containing an arbitrary constant) that represents all solutions of the ODE.
  - The general solution (and hence every particular solution) has an explicit expression.
  - Points where a solution is discontinuous can be found without solving the ODE (they are identified from the coefficients).
- A nonlinear first order ODE does not necessarily have any of the above properties.