Section 2.5: Autonomous equations and population dynamics

Recall the following.

- an **autonomous** DE has the form $\frac{dy}{dt} = f(y)$
- **equilibrium solutions**/ **equilibrium points** of an autonomous DE can be found by locating roots of $f(y) = 0$
- we can use equilibrium solutions to sketch solution curves

A more accurate sketch of solution curves can be drawn by using **concavity**.

If $\frac{dy}{dt} = f(y)$, then by the Chain Rule,

$$y'' = \frac{d}{dt}\frac{dy}{dt} = \frac{d}{dt}f(y) = \frac{df}{dy}\frac{dy}{dt} = \frac{df}{dy}f(y)$$

Thus: the graph of y as a function of t:

- is concave up $\iff y'' > 0 \iff f(y)$ and $f'(y)$ have same sign.
- is concave up $\iff y'' < 0 \iff f(y)$ and $f'(y)$ have opposite sign.
- has an inflection point if $y'' = 0$.

Exponential growth

Exponential growth = the rate of change of the population is proportional to the current population:

$$\frac{dy}{dt} = ry$$

where r is:

- the rate of growth if $r > 0$,
- the rate of decline if $r < 0$.

The solution is (by separation of variables):

$$y(t) = y_0 e^{rt}$$

where $y_0 = y(0)$ is the initial value.

Equilibrium solutions:
one equilibrium solution $y = 0$: unstable if $r > 0$ and stable if $r < 0$.

If $r = 0$ the population does not grow, it remains constant (equal to the initial value).

Logistic growth:

The rate of growth depends on the population and the resources (a small population can grow exponentially but as it gets larger limitations of resources reduce the rate of growth). Modify the constant rate of growth r to depend on y.

Logistic equation:

$$\frac{dy}{dt} = r(1 - \frac{y}{K})y$$

where:

- $r > 0$ intrinsic rate of growth,
- $K > 0$ saturation level or environmental carrying capacity.

The solution for $y(0) = y_0$ is (by separation of variables):

$$y(t) = \frac{y_0 K}{y_0 + (K - y_0)e^{-rt}}$$

Two equilibrium solutions: $y = 0$ (unstable) and $y = K$ (stable).

(a) Sketch f vs y, identify and classify the equilibrium points of y.

(b) For $y \in \mathbb{R}$ determine whether y is concave up or concave down.

(c) Use the information in parts (a), (b) to sketch a few integral curves of the DE.