Section 2.6: Exact differential equations

Example:

$$\underbrace{(2xy^2 + 2y)}_{M(x,y)} + \underbrace{(2x^2y + 2x)}_{N(x,y)} \frac{dy}{dx} = 0$$

Suppose we know a function $\psi(x, y)$ such that

$$\frac{\partial \psi}{\partial x} = M(x, y), \quad \frac{\partial \psi}{\partial y} = N(x, y)$$

Here: $\psi(x, y) = x^2y^2 + 2xy$.

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Then

$$\frac{d}{dx}\psi(x, y(x)) = \frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial y}\frac{dy}{dx} = M(x, y) + N(x, y)\frac{dy}{dx}$$

i.e. our initial DE is equivalent to: $\dfrac{d}{dx}\psi(x, y(x)) = 0$,

which has solution $\psi(x, y(x)) = C$, where C is a constant.

In our case, the solution to the original DE is: $x^2y^2 + 2xy = C$, where C is a constant.

Definition

The first order DE: $M(x, y) + N(x, y)\dfrac{dy}{dx} = 0$ is **exact** if

$$\frac{\partial M(x, y)}{\partial y} = \frac{\partial N(x, y)}{\partial x}$$

This is equivalent to the fact that there is a function $\psi(x, y)$ such that

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Recall that $\frac{\partial^2 \psi}{\partial x \partial y} = \frac{\partial^2 \psi}{\partial y \partial x}$.

If $\frac{\partial \psi}{\partial x} = M$ and $\frac{\partial \psi}{\partial y} = N$, then $\frac{\partial M}{\partial y} = \frac{\partial^2 \psi}{\partial y \partial x} = \frac{\partial^2 \psi}{\partial x \partial y} = \frac{\partial N}{\partial x}$.

Important: this condition suffices to guarantee that ψ exists.

How to find ψ*?*

To solve the first order *exact* DE $M(x,y) + N(x,y)\frac{dy}{dx} = 0$

- **Determine** $\psi(x,y)$ so that $\frac{\partial \psi}{\partial x} = M$ and $\frac{\partial \psi}{\partial y} = N$:

(a) $\frac{\partial \psi}{\partial x} = M$ means (by integrating with respect to x):

$$\psi(x,y) = \int M(x,y)\,dx + h(y)$$

(b) differentiate this formula for $\psi(x,y)$ with respect to y and use that $\frac{\partial \psi}{\partial y} = N(x,y)$:

$$\frac{\partial \psi}{\partial y} = \frac{\partial}{\partial y}\int M(x,y)\,dx + h'(y)$$

i.e.

$$h'(y) = N(x,y) - \frac{\partial}{\partial y}\int M(x,y)\,dx$$

Rem: the RHS is in fact a function of y only because the DE is exact, i.e. $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

(c) Integrate to get $h(y)$.

- The **solution** of the DE is $\psi(x, y(x)) = C$, where C is the constant of integration.

Example: $(2xy^2 + 2y) + (2x^2y + 2x)\,\frac{dy}{dx} = 0$