Section 2.7: Substitution methods

First order ODEs that we can transform by a change of variables into DEs we can solve:

1) Homogeneous DE $\xrightarrow{\textit{change of variable}}$ separable DE

2) Bernoulli equation $\xrightarrow{\textit{change of variable}}$ linear DE

Definition

A function $f(x, y)$ is said to be **homogeneous of degree** k if

$$f(\lambda x, \lambda y) = \lambda^k f(x, y) \quad \forall \lambda \in \mathbb{R}, \ \forall (x, y) \text{ in the domain of } f$$

Examples:

(a) $f(x, y) = x^2 + xy + y^2$ is homogeneous of degree 2.

(b) $f(x, y) = \dfrac{xy}{x^2 + y^2}$

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(b) $f(x,y) = \dfrac{xy}{x^2 + y^2}$ is homogeneous of degree 0.

(c) $f(x,y) = \dfrac{xy}{x^2 + y}$

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(c) $f(x, y) = \dfrac{xy}{x^2 + y}$ is not homogeneous.

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(c) $f(x,y) = \dfrac{xy}{x^2+y}$ is not homogeneous.

Suppose $f(x,y)$ is homogeneous of degree k.

For $x \neq 0$ we can factor $(x,y) = x\left(1, \frac{y}{x}\right)$. We get $f(x,y) = x^k f\left(1, \frac{y}{x}\right)$.

Example: $x^2 + xy + y^2 = x^2\left(1 + \frac{y}{x} + \left(\frac{y}{x}\right)^2\right)$.

Similarly, for $y \neq 0$ we can factor $(x,y) = y\left(\frac{x}{y}, 1\right)$. We get $f(x,y) = y^k f\left(\frac{x}{y}, 1\right)$.

Example: $x^2 + xy + y^2 = y^2\left(\left(\frac{x}{y}\right)^2 + \frac{x}{y} + 1\right)$.

First order homogeneous DE

Definition

A first order differential equation is called **homogeneous** if it is of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0$$

where M and N are homogeneous of the same degree.

Remark: a more appropriate name would be first order DE **with homogeneous coefficients** (not to be confused with the homogeneous linear DEs).

Substitution Method 1: For $x \neq 0$:

- rewrite the coefficients M and N as functions of $\frac{y}{x}$
- substitute $u = \frac{y}{x}$ i.e. $y = xu$. By the Chain Rule: $\frac{dy}{dx} = u + x\frac{du}{dx}$
- substitute in the DE and get a separable DE
- solve the equation by method of separable variables (section 2.1).

Example: $(x^2 + xy + y^2) - x^2\frac{dy}{dx}$

Substitution Method 2:

For $M(x,y) + N(x,y)\dfrac{dy}{dx} = 0$ and $y \neq 0$:

- replace $\dfrac{dy}{dx}$ with $\dfrac{dx}{dy}$ in the DE:
$$M(x,y)\frac{dx}{dy} + N(x,y) = 0$$
- rewrite the coefficients M and N as functions of $\dfrac{x}{y}$
- substitute $v = \dfrac{x}{y}$ i.e. $x = yv$. By the Chain Rule: $\dfrac{dx}{dy} = v + y\dfrac{dv}{dy}$
- substitute in the DE and get a separable DE
- solve the equation by method of separable variables (section 2.1).

Example: $(x^2 + xy + y^2)\dfrac{dy}{dx} - y^2 = 0$

Bernoulli differential equation

Definition

A **Bernoulli differential equation** is of the form

$$\frac{dy}{dt} + q(t)y = r(t)y^n$$

where n is any real number.

Remark: A Bernoulli DE is linear if and only if $n = 0$ or 1.

Example: $t^2 y' - ty + y^2 = 0$

Divide all terms by t^2 so that DE becomes: $y' - \frac{1}{t}y + \frac{1}{t^2}y^2 = 0$.

This is a Bernoulli DE, with $q(t) = -\frac{1}{t}$, $r(t) = -\frac{1}{t^2}$ et $n = 2$.

Substitution method:

The substitution $u = y^{1-n}$ reduces a Bernoulli DE to a linear equation.