EXAMPLE: $(x^2+xy+y^2)-x^2\dfrac{dy}{dx}=0$ for $x>0$

$$\left.\begin{array}{l} M(x,y)=x^2+xy+y^2 \\ N(x,y)=-x^2 \end{array}\right\}\text{ homogenous of degree 2}$$

- Pour $x\neq 0$: $x^2+xy+y^2=x^2\left(1+\frac{y}{x}+\left(\frac{y}{x}\right)^2\right)$
- The DE becomes: $\left(1+\frac{y}{x}+\left(\frac{y}{x}\right)^2\right)-\frac{dy}{dx}=0$
- Change of variable: $u=\frac{y}{x}$, i.e. $y=xu$

$$\frac{dy}{dx}=\frac{d}{dx}(xu)=u+x\frac{du}{dx}$$

CHAIN RULE: $y=f(u,x)$ where u regarded as function of x:

$$\frac{dy}{dx}=\frac{\partial f}{\partial x}\frac{dx}{dx}+\frac{\partial f}{\partial u}\frac{du}{dx}=\frac{\partial f}{\partial x}+\frac{\partial f}{\partial u}\frac{du}{dx}$$

- Substitute in the DE:

$$(1+\cancel{u}+u^2)-\left(\cancel{u}+x\frac{du}{dx}\right)=0$$

$$x\frac{du}{dx}-(1+u^2)=0 \qquad \text{SEPARABLE}$$

$$\frac{1}{1+u^2}\frac{du}{dx}=\frac{1}{x} \qquad \left[du=\frac{du}{dx}dx\right]$$

$$\int\frac{1}{1+u^2}du=\int\frac{1}{x}dx \quad \text{i.e.}\quad \arctan(u)=\ln|x|+C\underset{x>0}{=}\ln x+C$$

i.e. $u=\tan(\ln x+C)$, i.e. $\frac{y}{x}=\tan(\ln x+C)$

$$y=x\tan(\ln x+C)$$

EXAMPLE $(x^2+xy+y^2)\dfrac{dy}{dx}-y^2=0$ ← $y=0$ is a solution

Replace $\frac{dy}{dx}$ by $\frac{dx}{dy}$: $(x^2+xy+y^2)-y^2\frac{dx}{dy}=0$

The DE becomes $\left(\left(\frac{x}{y}\right)^2+\left(\frac{x}{y}\right)+1\right)^2-\frac{dx}{dy}=0$

Change of variable: $v=\frac{x}{y}$, i.e. $x=yv$

$$\frac{dx}{dy}=\frac{d}{dy}(yv)=v+y\frac{dv}{dy}$$

Substitute in the DE: $(v^2+\cancel{v}+1)^2-\left(\cancel{v}+y\frac{dv}{dy}\right)=0$

Continue as in the previous example.

EXAMPLE $y' - \frac{1}{t} y + \frac{1}{t^2} y^2 = 0$, $t>0$

Bernoulli diff. equation
$[y' + q(t) y = r(t) y^n$ with
$q(t) = -\frac{1}{t}$, $r(t) = -\frac{1}{t^2}$, $n=2]$

$y=0$ is a solution.
suppose now that $y \neq 0$:

$$\frac{1}{y^2} y' - \frac{1}{t}\frac{1}{y} + \frac{1}{t^2} = 0$$

change of variables: $u = \frac{1}{y}$

$$\frac{du}{dt} = -\frac{1}{y^2}\frac{dy}{dt}$$

$$\left[\begin{array}{l} u = y^{1-n} \\ u' = (1-n) y^{-n} y' \end{array}\right]$$

Substitution: $-u' - \frac{1}{t} u + \frac{1}{t^2} = 0$

$$(*) \qquad u' + \underbrace{\frac{1}{t}}_{p(t)} u = \frac{1}{t^2}$$ linear, 1st order

- $\int p(t) dt = \int \frac{1}{t} dt = \ln(t) + C$ (recall $t>0$): Can fix $C=0$ and choose $\mu(t) = e^{\ln(t)} = t$
- Multiply both sides of (*) by μ:

$$\underbrace{tu' + u}_{(tu)'} = \frac{1}{t} \quad \Rightarrow \quad tu = \int \frac{1}{t} dt + C = \ln(t) + C$$

$$\Rightarrow \quad \frac{t}{y} = \ln(t) + C$$

i.e. $$y = \frac{t}{\ln(t) + C}$$