Chapter 3: Systems of two first order DE

Section 3.1: Systems of two linear algebraic equations

Main topics:

- system of two equations
- matrix, determinant, trace and inverse
- solve systems with matrices
- eigenvalues and eigenvectors.

Systems of two linear equations

A **system of two linear equations** is of the form:

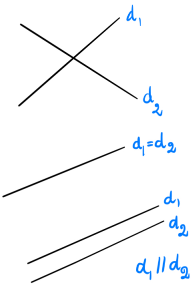
$$\begin{cases} a_{11}x_1 + a_{12}x_2 = b_1 \\ a_{21}x_1 + a_{22}x_2 = b_2 \end{cases}$$

where

- $a_{11}, a_{12}, a_{21}, a_{22}, b_1, b_2$ are fixed real numbers (the **coefficients** of the systems)
- x_1, x_2 are the unknowns.

Such a system either admits

- a unique solution,
- infinitely many solutions,
- no solution at all.



$d_1: a_{11}x_1 + a_{12}x_2 = b_1$

$d_2: a_{21}x_1 + a_{22}x_2 = b_2$

Unique intersection when the lines have different slopes:

$$-\frac{a_{11}}{a_{12}} \neq -\frac{a_{21}}{a_{22}}$$

i.e. $a_{11}a_{22} - a_{12}a_{21} \neq 0$

Matrix notation:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 = b_1 \\ a_{21}x_1 + a_{22}x_2 = b_2 \end{cases} \Leftrightarrow \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \Leftrightarrow Ax = b$$

Definition

$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ is the **matrix of coefficients** of the system

$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ is column vector of the unknowns (a 2×1 column vector)

$b = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$ is a given 2×1 column vector.

Definition

The system is said to be **homogeneous** if $b_1 = b_2 = 0$.

In matrix notation: $Ax = 0$ where on the RHS "0" means the zero vector $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

Determinant and trace of a 2×2 matrix

Let $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$.

Definition

The **determinant of** A, denoted by $\det(A)$, is the real number defined by

$$\det(A) = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{21}a_{12}\,.$$

The **trace of** A, denoted by $\mathrm{trace}(A)$, is the real number defined by

$$\mathrm{trace}(A) = a_{11} + a_{22}\,.$$

Definition

$I := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is the **identity matrix**.

We have $\det(I) = 1$ and $\mathrm{trace}(I) = 2$.

Moreover, one has $AI = IA = A$ for any matrix A.

Invertible matrices

Definition

We say that the matrix A is **invertible** or **non-singular** if $\det(A) \neq 0$.

[It is **noninvertible** or **singular** if $\det(A) = 0$.]

If the matrix A is invertible, then the **inverse** A^{-1} of A is the matrix uniquely defined by the formula:

$$A^{-1} = \frac{1}{\det(A)} \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}$$

The matrix A and its inverse A^{-1} are related by the property that:

$$AA^{-1} = A^{-1}A = I$$

Examples

- The determinant of $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ is $\det(A) = -2$. Hence A is invertible.

  The inverse of A is $A^{-1} = \frac{1}{-2}\begin{pmatrix} 4 & -2 \\ -3 & 1 \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{pmatrix}$.
- The determinant of $B = \begin{pmatrix} 4 & 2 \\ 2 & 1 \end{pmatrix}$ is $\det(B) = 0$. Hence B is noninvertible (or singular).

Theorem

The linear system

$$\begin{cases} a_{11}x_1 + a_{12}x_2 = b_1 \\ a_{21}x_1 + a_{22}x_2 = b_2 \end{cases}$$

admits a unique solution if and only if its associated matrix $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ *is invertible.*

In this case, the solution is given by $x = A^{-1}b$.

Indeed: $Ax = b \Leftrightarrow x = Ix = (A^{-1}A)x = A^{-1}(Ax) = A^{-1}b$.

Example

Suppose that $\det(A) \neq 0$: what is the (unique) solution of the homogeneous system $Ax = 0$?

Examples

- $$\begin{cases} 2x_1 - x_2 = 1 \\ x_1 + x_2 = 0 \end{cases}$$
  $A = \begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix}$, $\det(A) = 3 \neq 0$: unique solution
- $$\begin{cases} 2x_1 - x_2 = 1 \\ 2x_1 - x_2 = 1 \end{cases}$$
  $A = \begin{pmatrix} 2 & -1 \\ 2 & -1 \end{pmatrix}$, $\det(A) = 0$, equations multiples of each other: infinitely many solutions
- $$\begin{cases} 2x_1 - x_2 = 1 \\ 2x_1 - x_2 = 0 \end{cases}$$
  $A = \begin{pmatrix} 2 & -1 \\ 2 & -1 \end{pmatrix}$, $\det(A) = 0$, incompatible equations: no solution

Eigenvalues and eigenvectors

Consider a matrix $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$.

Definition

A (real or complex) number λ is said to be an **eigenvalue** of A if there exists a non-zero vector $v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ in $\mathbb{C}^2$ such that

$$Av = \lambda v\,.$$

In this case, v is called an **eigenvector** of A corresponding to the eigenvalue λ.

If λ is a real number we say that the eigenvalue is **real**.

How to find eigenvalues ?

If λ is an eigenvalue of A and $v(\neq 0)$ a corresponding eigenvector then one has:

$$Av = \lambda v \iff (A - \lambda I)v = 0$$

i.e. $(A - \lambda I)v = 0$ has a solution v which is a nonzero vector

i.e. $$\det(A - \lambda I) = 0.$$

Thus:
The eigenvalues of A are the numbers λ which are solutions of the equation

$$\det(A - \lambda I) = 0\,, \quad \text{called the } \textbf{characteristic equation} \text{ of } A.$$

Remark: we can always solve this equation because

$$\begin{aligned}
\det(A - \lambda I) &= \begin{vmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{vmatrix} \\
&= (a_{11} - \lambda)(a_{22} - \lambda) - a_{21}a_{12} \\
&= \lambda^2 - (a_{11} + a_{22})\lambda + (a_{11}a_{22} - a_{21}a_{12}) \\
&= \lambda^2 - \text{trace}(A)\lambda + \det(A)
\end{aligned}$$

is a polynomial of degree 2, called **characteristic polynomial** of A

Conclusion:
The eigenvalues of A are the **roots** of the **characteristic polynomial** $\det(A - \lambda I)$ of A, that is, the solutions the **characteristic equation** of A:

$$\det(A - \lambda I) = 0$$

Examples

- $A = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$ has two real eigenvalues
- $B = \begin{pmatrix} 3 & -2 \\ 4 & -1 \end{pmatrix}$ has two complex conjugate eigenvalues.

Remarks:

- Since A is a real matrix, its characteristic equation is a degree 2 equation with real coefficients (which are 1, $-\mathrm{trace}(A)$ and $\det(A)$).
- Consider the equation $\lambda^2 + b\lambda + c = 0$ where $b, c \in \mathbb{R}$. Its solutions λ_1, λ_2 are either both real numbers, or complex conjugate numbers.

How to find eigenvectors ?

- If v is an eigenvector for A for the eigenvalue λ, then $Av = \lambda v$, that is $(A - \lambda I)v = 0$.
- Solve the system of two linear algebraic equations $(A - \lambda I)v = 0$ where the coordinates of v are the unknowns.

Example $A = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$