Section 3.2: Systems of two first-order linear DE

Main topics:

- system of two first order linear differential equations
- matrix notation
- direction fields and phase portraits,
- examples of second order differential equations.

Definition

A **system of two first order linear differential equations** has the form:

$$\begin{cases} \dfrac{dx}{dt} = p_{11}(t)x + p_{12}(t)y + g_1(t) \\ \dfrac{dy}{dt} = p_{21}(t)x + p_{22}(t)y + g_2(t) \end{cases} \tag{S}$$

where

- $p_{11}, p_{12}, p_{21}, p_{22}$ and g_1, g_2 are given functions of t, defined on a same open interval I
- $x = x(t)$, $y = y(t)$ are **two** unknown functions of t (the **state variables**).

A **solution of (S)** consists of two differentiable functions $x = x(t)$, $y = y(t)$ satisfying (S) in some interval $I_0 \subseteq I$.

The system (S) and two initial conditions $x(t_0) = x_0$ and $y(t_0) = y_0$ form an **initial value problem (IVP)**.

Theorem (Theorem 3.2.1)

Suppose that the functions $p_{11}, p_{12}, p_{21}, p_{22}, g_1, g_2$ are continuous on an open interval I containing t_0. Then the IVP has a unique solution $x = x(t), y = y(t)$ on I.

System notation:

$$\begin{cases} \dfrac{dx}{dt} = p_{11}(t)x + p_{12}(t)y + g_1(t) \\ \dfrac{dy}{dt} = p_{21}(t)x + p_{22}(t)y + g_2(t) \end{cases}$$

Matrix notation:

$$\Leftrightarrow \quad \mathbf{X}'(t) = P(t)\mathbf{X}(t) + g(t)$$

Initial conditions:

$x(t_0) = x_0, y(t_0) = y_0 \qquad \Leftrightarrow \qquad \mathbf{X}(t_0) = \mathbf{X}_0$

where:

- $\mathbf{X}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$ = vector of unknown functions (the **state vector**).
- $\mathbf{X}'(t) = \dfrac{d\mathbf{X}}{dt} = \begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{pmatrix}$, $\underbrace{P(t) = \begin{pmatrix} p_{11}(t) & p_{12}(t) \\ p_{21}(t) & p_{22}(t) \end{pmatrix}}_{\text{matrix of coefficients}}$, $\underbrace{g(t) = \begin{pmatrix} g_1(t) \\ g_2(t) \end{pmatrix}}_{\text{the nonhomogeneous term, or input or forcing function}}$

  matrix of coefficients; the nonhomogeneous term, or **input** or **forcing function**
- $\mathbf{X}_0 = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$ = vector of initial conditions.

Definition

The system is called **homogeneous** if $g(t) = 0$ for all t (i.e. $g_1(t) = g_2(t) = 0$ for all t).

Example

Consider the IVP:

$$\begin{cases} \dfrac{dx}{dt} = tx - 3t^2 y + \sin(t) \\ \dfrac{dy}{dt} = \ln(t)x + \dfrac{1}{t} y - e^{2t-1} \end{cases}$$

with initial condition $x(1) = 1$ and $y(1) = 0$.

- In matrix notation, the IVP is:

$$\mathbf{X}'(t) = P(t)\mathbf{X}(t) + g(t)\,,$$

  where $P(t) = \begin{pmatrix} t & -3t^2 \\ \ln(t) & 1/t \end{pmatrix}$ and $g(t) = \begin{pmatrix} \sin(t) \\ -e^{2t-1} \end{pmatrix}$, with initial condition $\mathbf{X}(1) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

- The theorem of existence and uniqueness of solutions tells us that a solution $x = x(t), y = y(t)$ exists and is unique on the open interval

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- The theorem of existence and uniqueness of solutions tells us that a solution $x = x(t), y = y(t)$ exists and is unique on the open interval $(0, +\infty)$.

Plotting solutions

Definition

Let $\mathbf{X}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$ be a solution of the IVP

$$\mathbf{X}'(t) = P(t)\mathbf{X}(t) + g(t) \quad \text{with initial condition} \quad \mathbf{X}_0 = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}.$$

The graphs of the two functions $x = x(t)$ and $y = y(t)$ versus t are called **component plots**.

Plus: the component plot displays the dependence on t of a specific solution pair $x = x(t)$ and $y = y(t)$ (in particular their behaviour for very large t).
Minus: a new plot is needed if we change the initial conditions (and hence we change the solution).

More effective representations for *autonomous systems* (see next slide):

- trajectories (or orbits)
- direction fields
- phase portraits

Autonomous systems of two linear 1st order DE

Definition

A system of two linear differential equations $\mathbf{X}'(t) = P(t)\mathbf{X}(t) + g(t)$ is said to be **autonomous** if P and g are constant in t, i.e. it is of the form

$$\frac{d\mathbf{X}}{dt} = A\mathbf{X} + b$$

where:

- A is a 2×2 matrix with real coefficients
- b is a 2×1 column vector with real coefficients.

Remark: Recall that a 1st order DE is said to be autonomous if of the form $\frac{dy}{dt} = f(y)$, where f is constant in t. For a *linear* ODE, this means that $\frac{dy}{dt} = \alpha y + \beta$, where α, β are real numbers.

Example:

$$\begin{cases} \dfrac{dx}{dt} = x + y + 1 \\ \dfrac{dy}{dt} = 4x + y \end{cases}$$

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Example:

$$\begin{cases} \dfrac{dx}{dt} = x + y + 1 \\ \dfrac{dy}{dt} = 4x + y \end{cases} \quad \text{with} \quad A = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{is autonomous.}$$

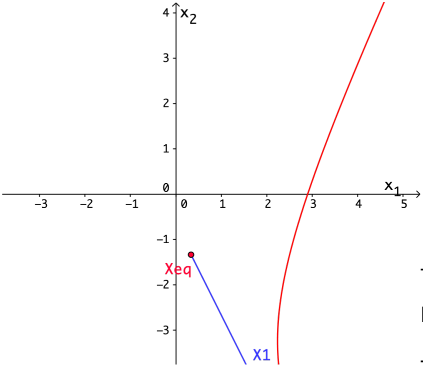
Consider the autonomous system of two 1st order linear DE:

$$\frac{d\mathbf{X}}{dt} = A\mathbf{X} + b \quad \text{where} \quad \mathbf{X}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} \quad \text{is the state vector}.$$

- The $x_1 x_2$-plane is called the **phase plane** (or **state plane)**.
- Let $x_1 = x_1(t), x_2 = x_2(t)$ be a solution. The curve $t \mapsto (x_1(t), x_2(t))$ in the phase plane is a **trajectory** (or **orbit**).
- A **direction field** is an array of vectors in the phase space: the vector $A\mathbf{X} + b$ vector is drawn with its tail at $\mathbf{X} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ for every choice de (x_1, x_2) is a fixed grid.
  If a trajectory passes through a point (x_1, x_2) of the grid, then its tangent vector at (x_1, x_2) is a vector of the direction field. Conversely, we can use a direction field to “guess" trajectories.
- An **equilibrium solution** is a solution for which $\frac{d\mathbf{X}}{dt} = 0$, i.e. $A\mathbf{X} + b = 0$.
  – If the matrix A is non singular, there is a unique equilibrium solution given by $\mathbf{X} = -A^{-1}b$. The equilibrium solution is a point in the phase plane.
  – If the matrix A is singular, there is either infinitely many equilibrium solutions or no equilibrium solution at all.
- A **phase portrait** is the plot a of representative sample of trajectories, including the equilibrium solutions, in the phase plane.

Example:

$$\begin{cases} \dfrac{dx_1}{dt} = x_1 + x_2 + 1 \\ \dfrac{dx_2}{dt} = 4x_1 + x_2 \end{cases} \quad \text{with} \quad A = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$



The phase plane $x_1 x_2$.

Equilibrium solution $X_{eq} = \begin{pmatrix} 1/3 \\ -4/3 \end{pmatrix}$

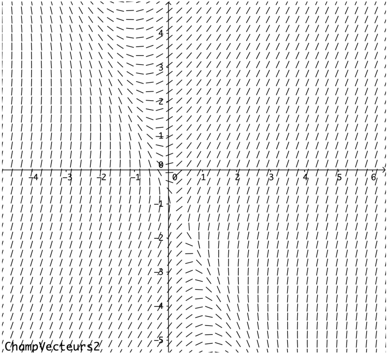
Trajectory for $X_1(t) = \begin{pmatrix} e^{-t} + 1/3 \\ -2e^{-t} - 4/3 \end{pmatrix}$

(with $\lim_{t \to +\infty} X_1(t) = X_{eq}$)

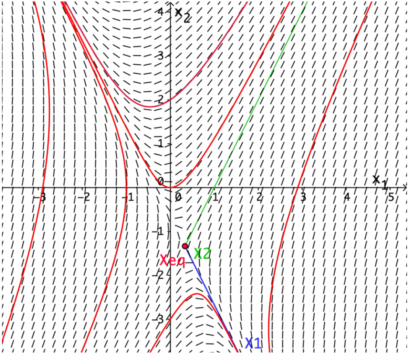
A third trajectory in red.

Example (continued):

$$\begin{cases} \dfrac{dx_1}{dt} = x_1 + x_2 + 1 \\ \dfrac{dx_2}{dt} = 4x_1 + x_2 \end{cases} \quad \text{with} \quad A = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$



Direction field in the phase plane $x_1 x_2$



Phase portrait with a few trajectories (including the equilibrium point X_{eq})

Applications to 2nd order DE

Consider the second order DE:

$$y'' + p(t)y' + q(t)y = g(t)$$

with initial conditions $y(t_0) = y_0$ and $y'(t_0) = y_1$.

- Set $x_1 = y$ and $x_2 = y'$. Then $x_1' = y' = x_2$ and $x_2' = y''$.
- The DE can now be rewritten as:

$$\begin{cases} x_2' + p(t)x_2 + q(t)x_1 = g(t) \\ x_1' = x_2 \end{cases}$$

  i.e.

$$\begin{cases} x_1' = x_2 \\ x_2' = -q(t)x_1 - p(t)x_2 + g(t) \end{cases}$$

  with initial conditions $x_1(t_0) = y_0$ and $x_2(t_0) = y_1$.
- Equivently, as the system of two first order DE's

$$\mathbf{X}' = \begin{pmatrix} 0 & 1 \\ -q(t) & -p(t) \end{pmatrix} \mathbf{X} + \begin{pmatrix} 0 \\ g(t) \end{pmatrix}$$

  with initial conditions $\mathbf{X}(t_0) = \begin{pmatrix} y_0 \\ y_1 \end{pmatrix}$

Example:

Transform the given DE equation into a system of first order equations:

$$u'' + 3tu' + 5u = t^2 + 4$$