(1) $\lambda_1 \neq \lambda_2$ both not zero

In this case, $\mathbf{x_{eq}} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is the unique equilibrium point (or critical point).

Recall: $\mathbf{x}(t) = C_1 e^{\lambda_1 t}\mathbf{v}_1 + C_2 e^{\lambda_2 t}\mathbf{v}_2$ is the general solution

▷ $\lambda_1 \neq \lambda_2$ and both negative, WLOG $\lambda_1 < \lambda_2 < 0$:

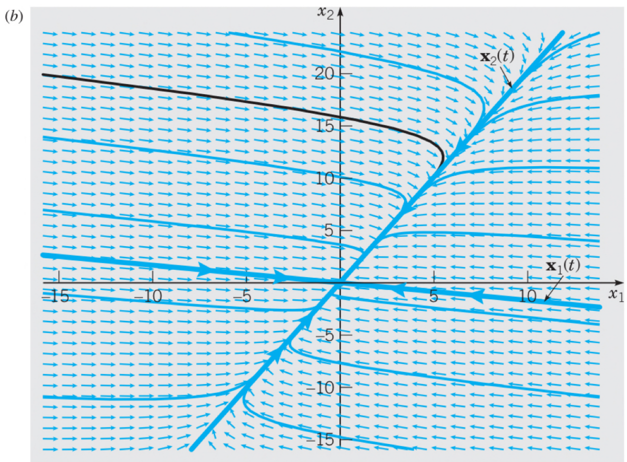
Then $\lim_{t\to+\infty} \mathbf{x}(t) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \mathbf{x_{eq}}$, i.e. all trajectories approach $\mathbf{x_{eq}}$ asymptotically as $t \to +\infty$: we say that $\mathbf{x_{eq}}$ is **asymptotically stable** and we call it a **nodal sink**.

If $C_2 \neq 0$, then
$$\mathbf{x}(t) = e^{\lambda_2 t}\left(C_1 e^{(\lambda_1-\lambda_2)t}\mathbf{v}_1 + C_2\mathbf{v}_2\right) \sim_{t\to+\infty} C_2 e^{\lambda_2 t}\mathbf{v}_2 = C_2\mathbf{x}_2(t)$$

i.e. all solutions with $C_2 \neq 0$ approach $\mathbf{x_{eq}}$ along the direction of $C_2\mathbf{v}_2$ (=the eigenvector with eigenvalue closest to 0).

If $C_1 \neq 0$, then
$$\mathbf{x}(t) = e^{\lambda_1 t}\left(C_1\mathbf{v}_1 + C_2 e^{(\lambda_2-\lambda_1)t}\mathbf{v}_2\right) \sim_{t\to-\infty} C_1 e^{\lambda_1 t}\mathbf{v}_1 = C_1\mathbf{x}_1(t)$$

i.e. all solutions with $C_1 \neq 0$ backward in time approach the direction of $C_1\mathbf{v}_1$.



J. BRENNAN et al., DIFFERENTIAL EQUATIONS, FIGURE 3.3.2

▷ $\lambda_1 \neq \lambda_2$ and both positive, WLOG $\lambda_1 > \lambda_2 > 0$:

Then all solutions diverge from $\mathbf{x_{eq}}$ for $t \to +\infty$ and $\lim_{t\to-\infty} \mathbf{x}(t) = \mathbf{x_{eq}}$, i.e. all trajectories approach $\mathbf{x_{eq}}$ backward in time $t \to -\infty$: we say that $\mathbf{x_{eq}}$ is **unstable** and we call it a **nodal source**.

$$\text{If } C_2 \neq 0, \text{ then} \quad \mathbf{x}(t) = e^{\lambda_2 t}\left(C_1 e^{(\lambda_1-\lambda_2)t}\mathbf{v}_1 + C_2\mathbf{v}_2\right) \quad \sim_{t\to-\infty} \quad C_2 e^{\lambda_2 t}\mathbf{v}_2 = C_2\mathbf{x}_2(t)$$

i.e. backward in time all solutions with $C_2 \neq 0$ approach $\mathbf{x_{eq}}$ along the direction of $C_2\mathbf{v}_2$ (=the eigenvector with eigenvalue closest to 0).

$$\text{If } C_1 \neq 0, \text{ then} \quad \mathbf{x}(t) = e^{\lambda_1 t}\left(C_1\mathbf{v}_1 + C_2 e^{(\lambda_2-\lambda_1)t}\mathbf{v}_2\right) \quad \sim_{t\to+\infty} \quad C_1 e^{\lambda_1 t}\mathbf{v}_1 = C_1\mathbf{x}_1(t)$$

i.e. all solutions with $C_1 \neq 0$ move for $t \to \infty$ asymptotically to a line of direction $C_1\mathbf{v}_1$.

(Picture as in the previous case "with arrows reversed")

Remark: In both cases, $\mathbf{x_{eq}}$ is approached (either in positive time or backward time) in a direction parallel to an eigenvector for the eigenvalue which is the closest to 0.

▷ $\lambda_1 \neq \lambda_2$ of opposite sign, WLOG $\lambda_2 < 0 < \lambda_1$:

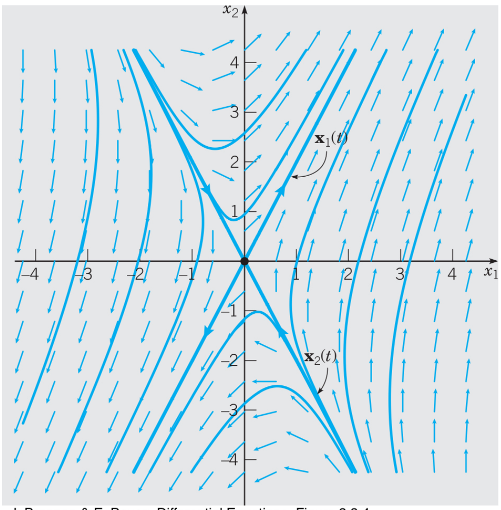
In this case there are solutions that tend to $\mathbf{x_{eq}}$ for $t \to +\infty$, but most of the solutions (those for $C_1 \neq 0$) grows to infinity: indeed, if $C_1 = 0$, then

$$\lim_{t \to +\infty} C_2 e^{\lambda_2 t} \mathbf{v}_2 = \mathbf{x_{eq}}$$

and if $C_1 \neq 0$, then

$$\mathbf{x}(t) = C_1 e^{\lambda_1 t} \mathbf{v}_1 + C_2 e^{\lambda_2 t} \mathbf{v}_2 \sim_{t \to +\infty} C_1 e^{\lambda_1 t} \mathbf{v}_1$$

grow to infinity in the direction of $C_1 \mathbf{v}_1$. We say that $\mathbf{x}_{\mathrm{eq}}$ is **unstable** and we call it a **saddle point**.



J. Brannan & E. Boyce, Differential Equations, Figure 3.3.4

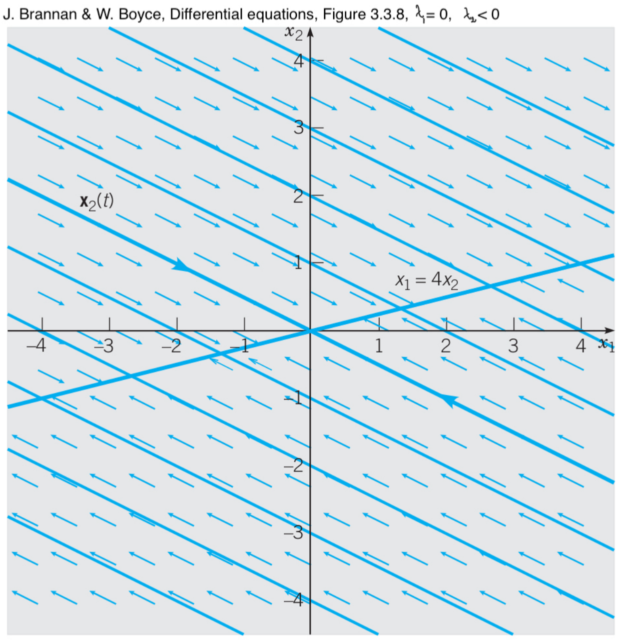
(2) $\lambda_1 \neq \lambda_2$ and $\lambda_1 = 0$

▷ $\lambda_1 = 0$ and $\lambda_2 < 0$

The general solution is $\mathbf{x}(t) = C_1\mathbf{x}_1(t) + C_2\mathbf{x}_2(t) = C_1\mathbf{v}_1 + C_2 e^{\lambda_2 t}\mathbf{v}_2$

Each eigenvector $\mathbf{v}$ of eigenvalue $\lambda_1 = 0$ is of the form $C_1\mathbf{v}_1$ and satisfies $A\mathbf{v} = 0$. So we have a line ℓ of critical points.

J. Brannan & W. Boyce, Differential equations, Figure 3.3.8, $\lambda_1 = 0$, $\lambda_2 < 0$



The trajectories $\mathbf{x}(t) = C_1\mathbf{v}_1 + C_2 e^{\lambda_2 t}\mathbf{v}_2$ passing through points not in ℓ (i.e. with $C_2 \neq 0$) move along half lines parallel to $C_2\mathbf{v}_2$ and asymptotically tend to the point on ℓ given by $\lim_{t\to+\infty} \mathbf{x}(t) = C_1\mathbf{v}_1$.

▷ $\lambda_1 = 0$ and $\lambda_2 > 0$

The situation is as above, with direction of the trajectories not passing though the critical line reversed since

$$\lim_{t\to-\infty} \mathbf{x}(t) = C_1\mathbf{v}_1 \,.$$

i.e. the trajectories passing through points which are not on L tend to points on L backward in time and diverge to infinity for $t \to +\infty$.