Section 3.3 (cont'd):

Recall: if $\mathbf{x}' = A\mathbf{x}$ is an autonomous (=constant coefficient) system of two linear DE, then its general solution is of the form

$$\mathbf{x}(t) = c_1\mathbf{x_1}(t) + c_2\mathbf{x_2}(t)$$

where $\mathbf{x_1}$ and $\mathbf{x_2}$ are two linearly independent solutions, c_1 and c_2 are constant.

How to find $\mathbf{x_1}$ *and* $\mathbf{x_2}$*?*

- The general solution of the linear DE $y' = ay$, where $a \in \mathbb{R}$, is $y(t) = Ce^{at}$.
- If $A = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}$ then the system $\mathbf{x}' = A\mathbf{x}$ is $\begin{cases} x_1' = a_1 x_1 \\ x_2' = a_2 x_2 \end{cases}$

  So $x_1(t) = e^{a_1 t}$ is a solution of the 1st DE and $\mathbf{x}(t) = \begin{pmatrix} x_1(t) \\ 0 \end{pmatrix} = e^{a_1 t}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is a solution of the system.

  Here a_1 is an eigenvalue of A and $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is an eigenvector with eigenvalue a_1.

Idea: We look for exponential solutions and use the eigenvalues of A.

- Suppose that $\mathbf{v}$ is eigenvector of A with eingenvalue λ, i.e. $A\mathbf{v} = \lambda\mathbf{v}$.
- $(e^{\lambda t}\mathbf{v})' = \lambda e^{t\lambda}\mathbf{v} = e^{\lambda t}(\lambda\mathbf{v}) = e^{\lambda t}(A\mathbf{v}) = A(e^{\lambda t}\mathbf{v})$,
- **Conclusion:** $\mathbf{x}(t) = e^{t\lambda}\mathbf{v}$ is a solution of the system $\mathbf{x}' = A\mathbf{x}$.

Example 1:

Consider the system of two linear differential equations $\mathbf{x}' = A\mathbf{x}$, where

$$A = \begin{pmatrix} 0 & 1 \\ -6 & -5 \end{pmatrix}.$$

- Find the eigenvalues λ_1 and λ_2 of A.
- Find the eigenvectors of A of eigenvalue λ_1 and those of eigenvalue λ_2.
- Fix an eigenvector $\mathbf{v}_1$ of eigenvalue λ_1 and verify that $\mathbf{x}_1(t) = e^{\lambda_1 t}\mathbf{v}_1$ is a solution of the system.
- Can you write the trace $\mathrm{trace}(A)$ of A and the determinant $\det(A)$ of A in terms of the eigenvalues λ_1 and λ_2?

The eigenvalues of A are solutions of the characteristic equation of A:

$$\det(A - \lambda I) = 0, \quad \text{i.e.} \quad \lambda^2 - \mathrm{trace}(A)\lambda - \det(A) = 0\,.$$

Properties of the characteristic equation:

- 2nd order equation in $\lambda \Rightarrow$ two (possibly equal) eigenvalues λ_1, λ_2
- real coefficients $\Rightarrow \begin{cases} \text{either both } \lambda_1, \lambda_2 \in \mathbb{R} \\ \text{or } \lambda_1, \lambda_2 \in \mathbb{C} \quad \text{and} \quad \lambda_2 = \overline{\lambda_1} \quad \text{(=complex conjugate of } \lambda_1\text{)} \end{cases}$
- $\mathrm{trace}(A) = \lambda_1 + \lambda_2 \quad$ and $\quad \det(A) = \lambda_1 \lambda_2$.

The form and dynamical properties of the solutions of the DE depend on cases:

I. both $\lambda_1, \lambda_2 \in \mathbb{R}$ and $\begin{cases} \text{I.a)} \quad \lambda_1 \neq \lambda_2 \quad \text{(see Section 3.3 below)} \\ \text{I.b)} \quad \lambda_1 = \lambda_2 \quad \text{(see Section 3.5)} \end{cases}$

II. $\lambda_1, \lambda_2 \in \mathbb{C}$ (not in $\mathbb{R}$) and $\lambda_2 = \overline{\lambda_1}$ (see Section 3.4)

For equilibrium (or critical) points $\mathbf{x_{eq}}$ (=solutions of $A\mathbf{x} = 0$) also need to distinguish:

- both λ_1, λ_2 are non-zero, i.e. $\det(A) \neq 0$, i.e. $\mathbf{x_{eq}} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ unique eq. point
- one of λ_1, λ_2 is zero, so $\det(A) = 0$, i.e. one straight line of eq. points $\mathbf{x_{eq}}$.

Case I.a): $\lambda_1, \lambda_2 \in \mathbb{R}$ and $\lambda_1 \neq \lambda_2$

$\mathbf{v}_1 = \begin{pmatrix} v_{11} \\ v_{21} \end{pmatrix}$ eigenvector of A of eigenvalue $\lambda_1 \Rightarrow \mathbf{x}_1(t) = e^{\lambda_1 t}\mathbf{v}_1$ solution of $\mathbf{x}' = A\mathbf{x}$.

$\mathbf{v}_2 = \begin{pmatrix} v_{12} \\ v_{22} \end{pmatrix}$ eigenvector of A of eigenvalue $\lambda_2 \Rightarrow \mathbf{x}_2(t) = e^{\lambda_2 t}\mathbf{v}_2$ solution of $\mathbf{x}' = A\mathbf{x}$.

Since $\lambda_1 \neq \lambda_2$, the solutions $\mathbf{x}_1$ and $\mathbf{x}_2$ are linearly independent.

Thus: the **general solution** of $\mathbf{x}' = A\mathbf{x}$ is

$$\mathbf{x}(t) = C_1\mathbf{x}_1(t) + C_2\mathbf{x}_2(t) = C_1 e^{\lambda_1 t}\mathbf{v}_1 + C_2 e^{\lambda_2 t}\mathbf{v}_2$$

Example. Write the general solution of the system of DE of Example 1.

For the dynamics of the solutions, we need to distinguish more cases:

(1) both λ_1, λ_2 are not zero → same sign → both positive / both negative; → opposite sign

(2) one of λ_1, λ_2 is zero

(3) $\lambda_1 = \lambda_2 = 0$

Remark: The case (3), i.e. $A = 0$, is degenerate: the system becomes $\mathbf{x}' = 0$, i.e. $\begin{cases} x_1' = 0 \\ x_2' = 0 \end{cases}$,

with general solution $\mathbf{x} = \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}$ (two constants). The cases to be studied are (1) and (2).

Example 1 (continued):

Consider the system of two linear differential equations $\mathbf{x}' = A\mathbf{x}$, where

$$A = \begin{pmatrix} 0 & 1 \\ -6 & -5 \end{pmatrix}.$$

We have found that the general solution is

$$\mathbf{x}(t) = C_1 e^{-3t} \begin{pmatrix} 1 \\ -3 \end{pmatrix} + C_2 e^{-2t} \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

- Determine the critical point(s) $\mathbf{x}_{\mathrm{eq}}$ of the system.
- Compute $\lim_{t\to+\infty} \mathbf{x}(t)$.
- Show that, if $C_2 \neq 0$, then

$$\mathbf{x}(t) \quad \sim_{t\to+\infty} \quad C_2 e^{-2t} \begin{pmatrix} 1 \\ -2 \end{pmatrix} = C_2 \mathbf{x}_2(t)$$

- Show that, if $C_1 \neq 0$, then

$$\mathbf{x}(t) \quad \sim_{t\to-\infty} \quad C_1 e^{-3t} \begin{pmatrix} 1 \\ -3 \end{pmatrix} = C_1 \mathbf{x}_1(t).$$