Section 3.3: Homogenous linear systems with constant coefficients

Main topics:

- Reduction to homogeneous systems of DE's,
- Homogenous systems:
  - ▷ Principle of superposition,
  - ▷ Wronskian and linear independance of solutions,
  - ▷ Use of eigenvalues and eigenvectors.

Reduction to homogeneous systems of DE's

Consider the system of DE's

$$\mathbf{x}' = A\mathbf{x} + b$$

where A is a 2×2 real matrix, b a 2×1 column vector.

Suppose A **is invertible**.

Then there is a unique equilibrium solution $\mathbf{x}_{\text{eq}} = -A^{-1}b$.

Change of variables: $\overline{\mathbf{x}} = \mathbf{x} - \mathbf{x}_{\text{eq}}$

Then

$$\overline{\mathbf{x}}' = (\mathbf{x} - \mathbf{x}_{\text{eq}})' = \mathbf{x}' = A\mathbf{x} + b = A(\mathbf{x} - (-A^{-1}b)) = A(\mathbf{x} - \mathbf{x}_{\text{eq}}) = A\overline{\mathbf{x}}$$

Therefore, the system reduces to the homogeneous system

$$\overline{\mathbf{x}}' = A\overline{\mathbf{x}}$$

Example:

Reduce the following system to a homogeneous one:

$$\begin{cases} \dfrac{dx}{dt} = x + y + 1 \\ \dfrac{dy}{dt} = 4x + y \end{cases} \quad \text{with} \quad A = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \mathbf{x}_{\text{eq}} = \begin{pmatrix} 1/3 \\ -4/3 \end{pmatrix}.$$

Homogeneous systems of two linear DEs (constant coefficients)

Theorem (Theorem 3.3.1)

Suppose that $\mathbf{x_1} = \mathbf{x_1}(t)$ *and* $\mathbf{x_2} = \mathbf{x_2}(t)$ *are solutions of the homogeneous system*

$$\mathbf{x}' = A\mathbf{x}$$

Then for any real (or complex) numbers c_1, c_2*,*

$$\mathbf{x}(t) = c_1\mathbf{x_1}(t) + c_2\mathbf{x_2}(t)$$

is also a solution of the system.

This theorem provides a tool to generate solutions from two fixed solutions.
It is known as the **principle of superposition**.

Definition

If $\mathbf{x}(t) = c_1\mathbf{x_1}(t) + c_2\mathbf{x_2}(t)$ for all t, then we say that $\mathbf{x}$ is a **linear combination** of $\mathbf{x_1}$ and $\mathbf{x_2}$, and we write it as: $\mathbf{x} = c_1\mathbf{x_1} + c_2\mathbf{x_2}$.

Let $\mathbf{x}_1(t) = \begin{pmatrix} x_{11}(t) \\ x_{21}(t) \end{pmatrix}$ and $\mathbf{x}_2(t) = \begin{pmatrix} x_{12}(t) \\ x_{22}(t) \end{pmatrix}$ be two solutions of the system $\mathbf{x}' = A\mathbf{x}$.

Definition

The **Wronskian** of $\mathbf{x}_1$ and $\mathbf{x}_2$ is the function $W[\mathbf{x}_1, \mathbf{x}_2]$ defined at t by the determinant

$$W[\mathbf{x}_1, \mathbf{x}_2](t) = \begin{vmatrix} x_{11}(t) & x_{12}(t) \\ x_{21}(t) & x_{22}(t) \end{vmatrix}$$

Definition

The solutions $\mathbf{x}_1$ and $\mathbf{x}_2$ of the system are said **linearly dependent** in an open interval I if there are constants c_1, c_2 (not both zero and *independent of* t) such that

$$c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t) = 0 \text{ for all } t \in I.$$

Two solutions that are not linearly dependent are called **linearly independent**.

$\mathbf{x}_1$ and $\mathbf{x}_2$ are linearly independent if and only if $W[\mathbf{x}_1, \mathbf{x}_2](t) \neq 0$ for all t in I.

Two solutions $\mathbf{x}_1$ and $\mathbf{x}_2$ that are linearly independent are said to form a **fundamental set of solutions**.

Remark: $\mathbf{x}_1$ and $\mathbf{x}_2$ are linearly dependent if and only if one is a constant multiple of the other: there is a constant k (independent of t) such that

$$\mathbf{x}_1(t) = k\mathbf{x}_2(t) \text{ for all } t \quad \textit{or} \quad \mathbf{x}_2(t) = k\mathbf{x}_1(t) \text{ for all } t.$$

Theorem (Theorem 3.3.4)

Suppose $\mathbf{x_1}$ and $\mathbf{x_2}$ for two linearly independent solutions of the system $\mathbf{x}' = A\mathbf{x}$. Then any solution of the above system is of the form

$$\mathbf{x}(t) = c_1\mathbf{x_1}(t) + c_2\mathbf{x_2}(t)$$

*for some constants c_1 and c_2. This is the **general solution** of the system.*
If, moreover, we fix an intitial condition $\mathbf{x}(t_0) = \mathbf{x}_0$, where $\mathbf{x}_0 = \begin{pmatrix} x_{10} \\ x_{20} \end{pmatrix}$ is a constant vector, then the constants c_1 and c_2 are uniquely determined and the solution to the system is unique.

Conclusion:

- The general solution of a homogenous system of **two** linear first order DEs is a linear combination of **two linearly independent** solutions (=one is not a multiple of the other)
- To find the general solution it is enough to find two linearly independent solutions.
- An initial condition uniquely determines the constants c_1 and c_2 and hence yields a unique solution to an IVP.

The eigenvalues and eigenvectors of the matrix coefficients A allow us to find the two independent solutions we need to solve $\mathbf{x}' = A\mathbf{x}$.

IDEA:

- The general solution of one linear homogenous DE $y' = ay$, where a is a constant, is $y(t) = Ce^{at}$.
- If $A = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}$ then the system $\mathbf{x}' = A\mathbf{x}$ is $\begin{cases} x_1' = a_1 x_1 \\ x_2' = a_2 x_2 \end{cases}$.

  So $x_1(t) = e^{a_1 t}$ is a solution of the 1st DE and $\mathbf{x}(t) = \begin{pmatrix} x_1(t) \\ 0 \end{pmatrix} = e^{a_1 t} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is a solution of the system.

  Here a_1 is an eigenvalue of A and $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is an eigenvector with eigenvalue a_1.

Conclusion: We look for exponential solutions and use the eigenvalues of A.

- Suppose that v is eigenvector of A with eingenvalue λ, i.e. $Av = \lambda v$.
- $(e^{\lambda t} v)' = \lambda e^{t\lambda} v$, so that $(e^{\lambda t} v)' = e^{\lambda t}(\lambda v) = A(e^{\lambda t} v)$,
- Then $\mathbf{x}(t) = e^{t\lambda} v$ is a solution of the system $\mathbf{x}' = A\mathbf{x}$.