$$\boxed{\mathbf{x}'(t) = A\,\mathbf{x}(t) \quad \text{where} \quad A = \begin{pmatrix} 0 & 1 \\ -6 & -5 \end{pmatrix}}$$

EXAMPLE 1
(section 3.3 - continued)

- Eigenvalues: roots of $\det(A-\lambda I)=0$, i.e. $\begin{vmatrix} -\lambda & 1 \\ -6 & -5-\lambda \end{vmatrix} = 0$

i.e. $\lambda(5+\lambda) + 6 = 0 \Rightarrow \lambda_1 = -3,\ \lambda_2 = -2$
$\lambda^2 + 5\lambda + 6 = 0$

- $\mathbf{v}_1 = \begin{pmatrix} v_{11} \\ v_{21} \end{pmatrix}$ is eigenvector with eigenvalue $\lambda_1 = -3$

$(A+3I)\mathbf{v}_1 = 0 \Leftrightarrow \begin{cases} 3v_{11} + v_{21} = 0 \\ -6v_{11} - 2v_{21} = 0 \end{cases} \Leftrightarrow 3v_{11} + v_{21} = 0 \Leftrightarrow v_{21} = -3v_{11}$

Since $\mathbf{v}_1 \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$: $\mathbf{v}_1 = \begin{pmatrix} c \\ -3c \end{pmatrix} = c\begin{pmatrix} 1 \\ -3 \end{pmatrix}$, c const $\neq 0$.

- $\mathbf{v}_2 = \begin{pmatrix} v_{12} \\ v_{22} \end{pmatrix}$ is eigenvector with eigenvalue $\lambda_2 = -2$

$(A+2I)\mathbf{v}_2 = 0 \Leftrightarrow \begin{cases} 2v_{12} + v_{22} = 0 \\ -6v_{12} - 3v_{22} = 0 \end{cases} \Leftrightarrow 2v_{12} + v_{22} = 0 \Leftrightarrow v_{22} = -2v_{12}$

Since $\mathbf{v}_2 \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$: $\mathbf{v}_2 = \begin{pmatrix} c \\ -2c \end{pmatrix} = c\begin{pmatrix} 1 \\ -2 \end{pmatrix}$, c const. $\neq 0$

- Check that $\mathbf{x}_1(t) = e^{-3t}\begin{pmatrix} 1 \\ -3 \end{pmatrix}$ is solution of $\mathbf{x}'(t) = A\,\mathbf{x}(t)$

$\mathbf{x}_1'(t) = -3e^{-3t}\begin{pmatrix} 1 \\ -3 \end{pmatrix}$ and

$A\,\mathbf{x}_1(t) = \begin{pmatrix} 0 & 1 \\ -6 & -5 \end{pmatrix} e^{-3t}\begin{pmatrix} 1 \\ -3 \end{pmatrix} = e^{-3t}\begin{pmatrix} 0 & 1 \\ -6 & -5 \end{pmatrix}\begin{pmatrix} 1 \\ -3 \end{pmatrix} = e^{-3t}\begin{pmatrix} -3 \\ 9 \end{pmatrix} = -3e^{-3t}\begin{pmatrix} 1 \\ -3 \end{pmatrix}$

Thus $\mathbf{x}_1'(t) = A\,\mathbf{x}(t)$

- General solution:

$$\mathbf{x}(t) = c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t) = c_1 e^{-3t}\begin{pmatrix} 1 \\ -3 \end{pmatrix} + c_2 e^{-2t}\begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

i.e. $\mathbf{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$ where $\begin{cases} x_1(t) = c_1 e^{-3t} + c_2 e^{-2t} \\ x_2(t) = -3c_1 e^{-3t} - 2c_2 e^{-2t} \end{cases}$

- Critical point(s) $\mathbf{x}_{eq}$ of $\mathbf{x}'(t) = A\mathbf{x}(t)$, $A = \begin{pmatrix} 0 & 1 \\ -6 & -5 \end{pmatrix}$

They are the solutions of $A\mathbf{x} = 0$. Since $\det(A) \neq 0$, there is a unique solution, namely $\mathbf{x}_{eq} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

Consider the general solution

$$\mathbf{x}(t) = C_1 \mathbf{x}_1(t) + C_2 \mathbf{x}_2(t) = C_1 e^{-3t}\begin{pmatrix} 1 \\ -3 \end{pmatrix} + C_2 e^{-2t}\begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

- Compute $\lim_{t\to+\infty} \mathbf{x}(t)$

Since $\lim_{t\to+\infty} e^{-at} = 0$ for $a>0$, we conclude

$$\lim_{t\to+\infty} \mathbf{x}(t) = 0 \cdot \begin{pmatrix} 1 \\ -3 \end{pmatrix} + 0 \cdot \begin{pmatrix} 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \mathbf{x}_{eq}$$

- Show that, if $C_2 \neq 0$, then

$$\mathbf{x}(t) \underset{t\to+\infty}{\sim} C_2 e^{-2t}\begin{pmatrix} 1 \\ -2 \end{pmatrix} = C_2 \mathbf{x}_2(t)$$

$$\mathbf{x}(t) = C_1 e^{-3t}\begin{pmatrix} 1 \\ -3 \end{pmatrix} + C_2 e^{-2t}\begin{pmatrix} 1 \\ -2 \end{pmatrix} = e^{-2t}\left(C_2\begin{pmatrix} 1 \\ -2 \end{pmatrix} + C_1 \underbrace{e^{(-3+2)t}}_{e^{-t} \to 0 \text{ for } t\to+\infty}\begin{pmatrix} 1 \\ -3 \end{pmatrix}\right)$$

$$\underset{t\to+\infty}{\sim} e^{-2t} C_2\begin{pmatrix} 1 \\ -2 \end{pmatrix} = C_2 \mathbf{x}_2(t)$$

- Show that, if $C_1 \neq 0$, then

$$\mathbf{x}(t) \underset{t\to-\infty}{\sim} C_1 e^{-3t}\begin{pmatrix} 1 \\ -3 \end{pmatrix} = C_1 \mathbf{x}_1(t)$$

$$\mathbf{x}(t) = C_1 e^{-3t}\begin{pmatrix} 1 \\ -3 \end{pmatrix} + C_2 e^{-2t}\begin{pmatrix} 1 \\ -2 \end{pmatrix} = e^{-3t}\left(C_1\begin{pmatrix} 1 \\ -3 \end{pmatrix} + C \underbrace{e^{(3-2)t}}_{e^{t} \to 0 \text{ for } t\to-\infty}\begin{pmatrix} 1 \\ -2 \end{pmatrix}\right)$$

$$\underset{t\to-\infty}{\sim} e^{-3t} C_1\begin{pmatrix} 1 \\ -3 \end{pmatrix} = C_1 \mathbf{x}_1(t)$$