Section 3.4: Complex eigenvalues

Case II: $\lambda_1 = \lambda_2 \in \mathbb{C}$ and $\lambda_2 = \overline{\lambda_1}$

To shorten the notation, **write λ instead of λ_1**.

So, we are supposing that A has two complex conjugate (and not real) eigenvalues:

$$\lambda = \mu + i\nu \quad \text{and} \quad \overline{\lambda} = \mu - i\nu$$

where μ, ν are real numbers.

In particular: λ and $\overline{\lambda}$ are distinct and non-zero.

- Eigenvectors $\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ associated with complex eigenvalues have usually complex components $v_1 = a_1 + ib_1$, $v_2 = a_2 + ib_2$ (with $a_1, b_1, a_2, b_2 \in \mathbb{R}$).
- If $\mathbf{v} = \mathbf{a} + i\mathbf{b} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} + i \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$ is an eigenvector of eigenvalue λ, i.e. $A\mathbf{v} = \lambda\mathbf{v}$ then $\overline{v} = \mathbf{a} - i\mathbf{b} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} - i \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$ is eigenvector of eigenvalue $\overline{\lambda}$, i.e. $A\overline{\mathbf{v}} = \overline{\lambda}\,\overline{\mathbf{v}}$.
- Two linearly independent solutions of $\mathbf{x}' = A\mathbf{x}$ are
$$\mathbf{x}_1(t) = e^{\lambda t}\mathbf{v} = e^{(\mu+i\nu)t}\mathbf{v} \quad \text{and} \quad \mathbf{x}_2(t) = e^{\overline{\lambda} t}\overline{\mathbf{v}} = e^{(\mu-i\nu)t}\overline{\mathbf{v}}$$

We have two linearly independent **complex-valued** solutions of $\mathbf{x}' = A\mathbf{x}$, namely

$$\mathbf{x}_1(t) = e^{\lambda t}\mathbf{v} = e^{(\mu+i\nu)t}\mathbf{v} \qquad \text{and} \qquad \mathbf{x}_2(t) = e^{\overline{\lambda}t}\overline{\mathbf{v}} = e^{(\mu-i\nu)t}\overline{\mathbf{v}}$$

We want to have two **real-valued** solutions.

- $\mathbf{x}_2(t) = \overline{\mathbf{x}_1(t)}$ [because $\overline{z}\,\overline{s} = \overline{zs}$ for $z, s \in \mathbb{C}$].
- Linear combinations of solutions are solutions (principle of superposition): since $\mathbf{x}_1$ and $\mathbf{x}_2 = \overline{\mathbf{x}_1}$ are solutions, so are

$$\mathbf{u} = \frac{1}{2}\mathbf{x}_1 + \frac{1}{2}\mathbf{x}_2 = \frac{\mathbf{x}_1 + \overline{\mathbf{x}_1}}{2} = \mathrm{Re}\,\mathbf{x}_1$$

and

$$\mathbf{w} = \frac{1}{2i}\mathbf{x}_1 - \frac{1}{2i}\mathbf{x}_2 = \frac{\mathbf{x}_1 - \overline{\mathbf{x}_1}}{2i} = \mathrm{Im}\,\mathbf{x}_1$$

- $\mathbf{u} = \mathrm{Re}\,\mathbf{x}_1$ and $\mathbf{w} = \mathrm{Im}\,\mathbf{x}_1$ are real-valued solutions.
- **Fact:** **u** and **w** are linearly-independent.

Conclusion: The general solution of $\mathbf{x}' = A\mathbf{x}$ is:

$$\mathbf{x}(t) = C_1\mathbf{u}(t) + C_2\mathbf{w}(t)$$

where C_1, C_2 are constants.

Example:

- Determine the general solution of $\mathbf{x}' = A\mathbf{x}$ where $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$
- Find the solution of the IVP for $\mathbf{x}' = A\mathbf{x}$ with initial condition $\mathbf{x}(0) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$.