Section 3.4: Complex eigenvalues

Case II: $\lambda_1 = \lambda_2 \in \mathbb{C}$ and $\lambda_2 = \overline{\lambda_1}$

To shorten the notation, **write λ instead of λ_1**.

So, we are supposing that A has two complex conjugate (and not real) eigenvalues:

$$\lambda = \mu + i\nu \quad \text{and} \quad \overline{\lambda} = \mu - i\nu$$

where μ, ν are real numbers.

In particular: λ and $\overline{\lambda}$ are distinct and non-zero.

- Eigenvectors $\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ associated with complex eigenvalues have usually complex components $v_1 = a_1 + ib_1$, $v_2 = a_2 + ib_2$ (with $a_1, b_1, a_2, b_2 \in \mathbb{R}$).
- If $\mathbf{v} = \mathbf{a} + i\mathbf{b} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} + i \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$ is an eigenvector of eigenvalue λ, i.e. $A\mathbf{v} = \lambda\mathbf{v}$

  then $\overline{v} = \mathbf{a} - i\mathbf{b} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} - i \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$ is eigenvector of eigenvalue $\overline{\lambda}$, i.e. $A\overline{\mathbf{v}} = \overline{\lambda}\,\overline{\mathbf{v}}$.
- Two linearly independent solutions of $\mathbf{x}' = A\mathbf{x}$ are

$$\mathbf{x}_1(t) = e^{\lambda t}\mathbf{v} = e^{(\mu+i\nu)t}\mathbf{v} \quad \text{and} \quad \mathbf{x}_2(t) = e^{\overline{\lambda} t}\overline{\mathbf{v}} = e^{(\mu-i\nu)t}\overline{\mathbf{v}}$$

We have two linearly independent **complex-valued** solutions of $\mathbf{x}' = A\mathbf{x}$, namely

$$\mathbf{x}_1(t) = e^{\lambda t}\mathbf{v} = e^{(\mu+i\nu)t}\mathbf{v} \qquad \text{and} \qquad \mathbf{x}_2(t) = e^{\overline{\lambda}t}\overline{\mathbf{v}} = e^{(\mu-i\nu)t}\overline{\mathbf{v}}$$

We want to have two **real-valued** solutions.

- $\mathbf{x}_2(t) = \overline{\mathbf{x}_1(t)}$ [because $\overline{z}\,\overline{s} = \overline{zs}$ for $z, s \in \mathbb{C}$].
- Linear combinations of solutions are solutions (principle of superposition): since $\mathbf{x}_1$ and $\mathbf{x}_2 = \overline{\mathbf{x}_1}$ are solutions, so are

$$\mathbf{u} = \frac{1}{2}\mathbf{x}_1 + \frac{1}{2}\mathbf{x}_2 = \frac{\mathbf{x}_1 + \overline{\mathbf{x}_1}}{2} = \operatorname{Re}\mathbf{x}_1$$

  and

$$\mathbf{w} = \frac{1}{2i}\mathbf{x}_1 - \frac{1}{2i}\mathbf{x}_2 = \frac{\mathbf{x}_1 - \overline{\mathbf{x}_1}}{2i} = \operatorname{Im}\mathbf{x}_1$$

- $\mathbf{u} = \operatorname{Re}\mathbf{x}_1$ and $\mathbf{w} = \operatorname{Im}\mathbf{x}_1$ are real-valued solutions.
- **Fact:** $\mathbf{u}$ and $\mathbf{w}$ are linearly-independent.

Conclusion: The general solution of $\mathbf{x}' = A\mathbf{x}$ is:

$$\mathbf{x}(t) = C_1\mathbf{u}(t) + C_2\mathbf{w}(t)$$

where C_1, C_2 are constants.

Example:

- Determine the general solution of $\mathbf{x}' = A\mathbf{x}$ where $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

Example:

- Determine the general solution of $\mathbf{x}' = A\mathbf{x}$ where $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

  General solution: $\mathbf{x}(t) = C_1 \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix} + C_2 \begin{pmatrix} \sin t \\ \cos t \end{pmatrix}$, where C_1, C_2 are constants and $t \in \mathbb{R}$.

- Find the solution of the IVP for $\mathbf{x}' = A\mathbf{x}$ with initial condition $\mathbf{x}(0) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

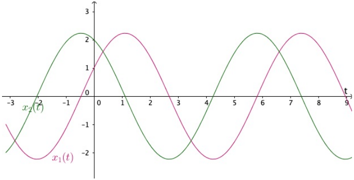
Example:

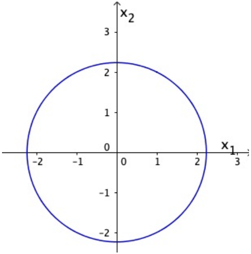
- Determine the general solution of $\mathbf{x}' = A\mathbf{x}$ where $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

  General solution: $\mathbf{x}(t) = C_1 \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix} + C_2 \begin{pmatrix} \sin t \\ \cos t \end{pmatrix}$, where C_1, C_2 are constants and $t \in \mathbb{R}$.

- Find the solution of the IVP for $\mathbf{x}' = A\mathbf{x}$ with initial condition $\mathbf{x}(0) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

$x_1(t) = \cos t + 2\sin t$
$x_2(t) = -\sin t + 2\cos t$





The trajectory in the phase plane is a circle.

Explicit expressions of u and w (general case):

- $\mathbf{u}(t) = \mathrm{Re}\,\mathbf{x}_1(t) = \mathrm{Re}\left(e^{(\mu+i\nu)t}\mathbf{v}\right)$

  and

  $\mathbf{w}(t) = \mathrm{Im}\,\mathbf{x}_1(t) = \mathrm{Im}\left(e^{(\mu+i\nu)t}\mathbf{v}\right)$

Explicit expressions of u and w (general case):

- $\mathbf{u}(t) = \operatorname{Re} \mathbf{x}_1(t) = \operatorname{Re}\left(e^{(\mu+i\nu)t}\mathbf{v}\right)$

  and

  $\mathbf{w}(t) = \operatorname{Im} \mathbf{x}_1(t) = \operatorname{Im}\left(e^{(\mu+i\nu)t}\mathbf{v}\right)$

For $z, s \in \mathbb{C}$:	$\operatorname{Re}(zs) = \operatorname{Re} z \operatorname{Re} s - \operatorname{Im} z \operatorname{Im} s$ $\operatorname{Im}(zs) = \operatorname{Re} z \operatorname{Im} s + \operatorname{Re} z \operatorname{Im} s$

Explicit expressions of u and w (general case):

- $\mathbf{u}(t) = \operatorname{Re} \mathbf{x}_1(t) = \operatorname{Re}\left(e^{(\mu+i\nu)t}\mathbf{v}\right)$
  $$= \operatorname{Re} e^{(\mu+i\nu)t} \operatorname{Re} \mathbf{v} - \operatorname{Im} e^{(\mu+i\nu)t} \operatorname{Im} \mathbf{v}$$
  and
  $$\mathbf{w}(t) = \operatorname{Im} \mathbf{x}_1(t) = \operatorname{Im}\left(e^{(\mu+i\nu)t}\mathbf{v}\right)$$

For $z, s \in \mathbb{C}$:
$$\operatorname{Re}(zs) = \operatorname{Re} z \operatorname{Re} s - \operatorname{Im} z \operatorname{Im} s$$
$$\operatorname{Im}(zs) = \operatorname{Re} z \operatorname{Im} s + \operatorname{Re} z \operatorname{Im} s$$

Explicit expressions of u and w (general case):

- $$\mathbf{u}(t) = \operatorname{Re}\mathbf{x}_1(t) = \operatorname{Re}\left(e^{(\mu+i\nu)t}\mathbf{v}\right) = \operatorname{Re} e^{(\mu+i\nu)t}\operatorname{Re}\mathbf{v} - \operatorname{Im} e^{(\mu+i\nu)t}\operatorname{Im}\mathbf{v}$$

  and

  $$\mathbf{w}(t) = \operatorname{Im}\mathbf{x}_1(t) = \operatorname{Im}\left(e^{(\mu+i\nu)t}\mathbf{v}\right) = \operatorname{Re} e^{(\mu+i\nu)t}\operatorname{Im}\mathbf{v} + \operatorname{Im} e^{(\mu+i\nu)t}\operatorname{Re}\mathbf{v}$$

  For $z, s \in \mathbb{C}$:
  $$\operatorname{Re}(zs) = \operatorname{Re} z \operatorname{Re} s - \operatorname{Im} z \operatorname{Im} s$$
  $$\operatorname{Im}(zs) = \operatorname{Re} z \operatorname{Im} s + \operatorname{Re} z \operatorname{Im} s$$

Explicit expressions of u and w (general case):

- $$\mathbf{u}(t) = \mathrm{Re}\,\mathbf{x}_1(t) = \mathrm{Re}\left(e^{(\mu+i\nu)t}\mathbf{v}\right) = \mathrm{Re}\,e^{(\mu+i\nu)t}\,\mathrm{Re}\,\mathbf{v} - \mathrm{Im}\,e^{(\mu+i\nu)t}\,\mathrm{Im}\,\mathbf{v}$$

  and

  $$\mathbf{w}(t) = \mathrm{Im}\,\mathbf{x}_1(t) = \mathrm{Im}\left(e^{(\mu+i\nu)t}\mathbf{v}\right) = \mathrm{Re}\,e^{(\mu+i\nu)t}\,\mathrm{Im}\,\mathbf{v} + \mathrm{Im}\,e^{(\mu+i\nu)t}\,\mathrm{Re}\,\mathbf{v}$$

  For $z, s \in \mathbb{C}$:

  $$\mathrm{Re}(zs) = \mathrm{Re}\,z\,\mathrm{Re}\,s - \mathrm{Im}\,z\,\mathrm{Im}\,s$$
  $$\mathrm{Im}(zs) = \mathrm{Re}\,z\,\mathrm{Im}\,s + \mathrm{Re}\,z\,\mathrm{Im}\,s$$

- For $\mu, \nu, t \in \mathbb{R}$: $$e^{(\mu+i\nu)t} = e^{\mu t}e^{i\nu t} = e^{\mu t}\big(\cos(\nu t) + i\sin(\nu t)\big).$$

  Hence:

  $$\mathrm{Re}\,e^{(\mu+i\nu)t} = e^{\mu t}\cos(\nu t)$$
  $$\mathrm{Im}\,e^{(\mu+i\nu)t} = e^{\mu t}\sin(\nu t)\,.$$

Explicit expressions of u and w (general case):

- $$\mathbf{u}(t) = \operatorname{Re}\mathbf{x}_1(t) = \operatorname{Re}\left(e^{(\mu+i\nu)t}\mathbf{v}\right) = \operatorname{Re} e^{(\mu+i\nu)t}\operatorname{Re}\mathbf{v} - \operatorname{Im} e^{(\mu+i\nu)t}\operatorname{Im}\mathbf{v}$$

  and

  $$\mathbf{w}(t) = \operatorname{Im}\mathbf{x}_1(t) = \operatorname{Im}\left(e^{(\mu+i\nu)t}\mathbf{v}\right) = \operatorname{Re} e^{(\mu+i\nu)t}\operatorname{Im}\mathbf{v} + \operatorname{Im} e^{(\mu+i\nu)t}\operatorname{Re}\mathbf{v}$$

For $z, s \in \mathbb{C}$:	$\operatorname{Re}(zs) = \operatorname{Re} z\operatorname{Re} s - \operatorname{Im} z\operatorname{Im} s$
	$\operatorname{Im}(zs) = \operatorname{Re} z\operatorname{Im} s + \operatorname{Re} z\operatorname{Im} s$

- For $\mu, \nu, t \in \mathbb{R}$: $\quad e^{(\mu+i\nu)t} = e^{\mu t}e^{i\nu t} = e^{\mu t}\left(\cos(\nu t) + i\sin(\nu t)\right).$

  Hence: $\quad \operatorname{Re} e^{(\mu+i\nu)t} = e^{\mu t}\cos(\nu t)$

  $\qquad\qquad \operatorname{Im} e^{(\mu+i\nu)t} = e^{\mu t}\sin(\nu t)\,.$

- For $\mathbf{v} = \mathbf{a} + i\mathbf{b} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} + i\begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$, we have $\operatorname{Re}\mathbf{v} = \mathbf{a}$ and $\operatorname{Im}\mathbf{v} = \mathbf{b}$.

Explicit expressions of u and w (general case):

- $$\mathbf{u}(t) = \operatorname{Re}\mathbf{x}_1(t) = \operatorname{Re}\left(e^{(\mu+i\nu)t}\mathbf{v}\right) = \operatorname{Re} e^{(\mu+i\nu)t}\operatorname{Re}\mathbf{v} - \operatorname{Im} e^{(\mu+i\nu)t}\operatorname{Im}\mathbf{v}$$

  and

  $$\mathbf{w}(t) = \operatorname{Im}\mathbf{x}_1(t) = \operatorname{Im}\left(e^{(\mu+i\nu)t}\mathbf{v}\right) = \operatorname{Re} e^{(\mu+i\nu)t}\operatorname{Im}\mathbf{v} + \operatorname{Im} e^{(\mu+i\nu)t}\operatorname{Re}\mathbf{v}$$

  For $z, s \in \mathbb{C}$: $\operatorname{Re}(zs) = \operatorname{Re} z \operatorname{Re} s - \operatorname{Im} z \operatorname{Im} s$
  $\operatorname{Im}(zs) = \operatorname{Re} z \operatorname{Im} s + \operatorname{Re} z \operatorname{Im} s$

- For $\mu, \nu, t \in \mathbb{R}$: $e^{(\mu+i\nu)t} = e^{\mu t}e^{i\nu t} = e^{\mu t}\big(\cos(\nu t) + i\sin(\nu t)\big)$.

  Hence: $\operatorname{Re} e^{(\mu+i\nu)t} = e^{\mu t}\cos(\nu t)$
  $\operatorname{Im} e^{(\mu+i\nu)t} = e^{\mu t}\sin(\nu t)$.

- For $\mathbf{v} = \mathbf{a} + i\mathbf{b} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} + i\begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$, we have $\operatorname{Re}\mathbf{v} = \mathbf{a}$ and $\operatorname{Im}\mathbf{v} = \mathbf{b}$.

Conclusion:

$$\mathbf{u}(t) = e^{\mu t}[\cos(\nu t)\mathbf{a} - \sin(\nu t)\mathbf{b}]$$
$$\mathbf{w}(t) = e^{\mu t}[\cos(\nu t)\mathbf{b} + \sin(\nu t)\mathbf{a}].$$

Behavior of the solutions

Since $\det(A) = \lambda\overline{\lambda} \neq 0$

$$\mathbf{x}_{\text{eq}} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

is the unique equilibrium (or critical) point.
It corresponds to the origin $(0, 0)$ of the phase plane

The solutions

$$\mathbf{u}(t) = e^{\mu t}[\cos(\nu t)\mathbf{a} - \sin(\nu t)\mathbf{b}]$$
$$\mathbf{w}(t) = e^{\mu t}[\cos(\nu t)\mathbf{b} + \sin(\nu t)\mathbf{a}]\,.$$

have an oscillatory behavior as functions of t.

The nature of the oscillation depends on $\mu = \mathrm{Re}\,\lambda$.

$$\mathbf{u}(t) = e^{\mu t}[\cos(\nu t)\mathbf{a} - \sin(\nu t)\mathbf{b}]$$
$$\mathbf{w}(t) = e^{\mu t}[\cos(\nu t)\mathbf{b} + \sin(\nu t)\mathbf{a}] .$$

- **$\mu = 0$, i.e. $\lambda = i\nu$:**

$\mathbf{u}(t)$ and $\mathbf{v}(t)$ are periodic function of t of period $T = \frac{2\pi}{\nu}$.
In the phase plane:
The trajectories are ellipses.
The origin $(0, 0)$ is a called a **center** and it is said to be **stable**.

- $\mu < 0$:

As functions of t, the amplitude of the oscillations of $\mathbf{u}(t)$ and $\mathbf{v}(t)$ decays exponentially.
In the phase plane:
The trajectories spiral around $(0, 0)$ and approach $(0, 0)$ at $t \to +\infty$.
The origin $(0, 0)$ is a called a **spiral sink** and it is said to be **asymptotically stable**.

- $\mu > 0$:

As functions of t, the amplitude of the oscillations of $\mathbf{u}(t)$ and $\mathbf{v}(t)$ grows exponentially.
In the phase plane:
The trajectories spiral out of $(0, 0)$ as t increases.
The origin $(0, 0)$ is a called a **spiral source** and it is said to be **unstable**.