Section 3.5: Repeated eigenvalues

This is what we called Case Ia): two (necessarily real) equal eigenvalues $\lambda_1 = \lambda_2$ of A.
To shorten the notation, **write λ instead of λ_1**.
We suppose $\lambda \neq 0$.

A matrix A with two repeated eigenvalues can have one or two linearly independent eigenvectors.
The form and behavior of the solutions of $\mathbf{x}' = A\mathbf{x}$ is different according to these two situations.

Example:

Show that $A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}$ have one repeated eigenvalue λ.
Find λ.
Show that A has two linearly independent eigenvectors of eigenvalue λ whereas B does not.

Consider the system $\mathbf{x}' = A\mathbf{x}$ where A is a 2×2 matrix with repeted eigenvalue $\lambda \neq 0$.

- If there are two linearly independent eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_1$ of eigenvalue λ.
  Then two linearly independent solutions of $\mathbf{x}' = A\mathbf{x}$ are

  $$\mathbf{x}_1(t) = e^{\lambda t}\mathbf{v}_1 \qquad \text{and} \qquad \mathbf{x}_2(t) = e^{\lambda t}\mathbf{v}_2$$

  The **general solution** is

  $$\mathbf{x}(t) = C_1\mathbf{x}_1(t) + C_2\mathbf{x}_2(t) = C_1 e^{\lambda t}\mathbf{v}_1 + C_2 e^{\lambda t}\mathbf{v}_2.$$

- If there is only one linearly independent eigenvector $\mathbf{v}_1$ of eigenvalue λ.
  Then two linearly independent solutions of $\mathbf{x}' = A\mathbf{x}$ are

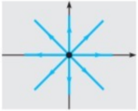
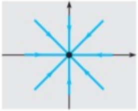
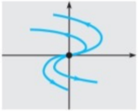
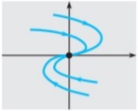
  $$\mathbf{x}_1(t) = te^{\lambda t}\mathbf{v}_1 \qquad \text{and} \qquad \mathbf{x}_2(t) = e^{\lambda t}\mathbf{w}$$

  where $\mathbf{w}$ satisfies $(A - \lambda I)\mathbf{w} = \mathbf{v}_1$
  (we say that $\mathbf{w}$ is a *generalized eigenvector* corresponding to the eigenvalue λ).
  The **general solution** is

  $$\mathbf{x}(t) = C_1\mathbf{x}_1(t) + C_2\mathbf{x}_2(t) = C_1 e^{\lambda t}\mathbf{v}_1 + C_2 te^{\lambda t}\mathbf{w}\,.$$

$\mathbf{x}_{\text{eq}} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is the unique critical point. It is the origin $(0,0)$ of the phase plane.

TABLE 3.5.1 Phase portraits for $\mathbf{x}' = \mathbf{A}\mathbf{x}$ when $\mathbf{A}$ has a single repeated eigenvalue.

Nature of **A** and Eigenvalues	Sample Phase Portrait	Type of Critical Point	Stability
$\mathbf{A} = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$ $\lambda > 0$		(0, 0) is an **unstable proper node**. *Note*: (0, 0) is also called an **unstable star node**.	Unstable
$\mathbf{A} = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$ $\lambda < 0$		(0, 0) is a **stable proper node**. *Note*: (0, 0) is also called a **stable star node**.	Asymptotically stable
A is not diagonal. $\lambda > 0$		(0, 0) is an **unstable improper node**. *Note*: (0, 0) is also called an **unstable degenerate node**.	Unstable
A is not diagonal. $\lambda < 0$		(0, 0) is a **stable improper node**. *Note*: (0, 0) is also called a **stable degenerate node**.	Asymptotically stable

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