# Chapter 4: Second Order Linear Equations

## 4.1: Definitions and Examples

### Definition

A second order differential equation is said to be **linear** if it can be written in **standard form**

$$y'' + p(t)y' + q(t)y = g(t)$$

where $p$, $q$ and $g$ are arbitrary functions of the independent variable $t$.

If the function is $g$ identically zero, then the equation is said to be **homogeneous**. It is said to be **nonhomogeneous** otherwise.

The equation is said to be a **constant coefficient equation** if the functions $p$, $q$ and $g$ are constant (i.e. do not depend on $t$). Otherwise, it is said to have **variable coefficients**.

# Example: the spring-mass system

A mass of magnitude $m$ is attached to the lower end of a spring of natural lenght $l$.
The mass causes an extension $L$ of the spring.
The spring has lenght $l + L$ at the equilibrium position.

**Goal:** to study the motion of the mass (along a vertical line) caused by external forces (the vibration problem).

**Model:**

vertical $y$-axis, positive direction downward, origin at the equilibrium position.
$y(t)$= position of the mass at time $t$
$y'(t)$= velocity of the mass at time $t$
Equilibrium position: $y(0) = 0$.

**Newton's law of motion:**

$$my''(t) = F_{\text{net}}(t)$$

where:
$y''(t)$= acceleration of the mass at time $t$
$F_{\text{net}}(t)$= the net force acting on the mass, with different components.

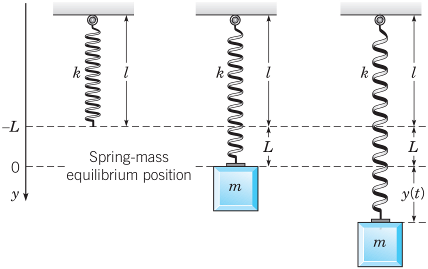


Figure 4.3.1 in: J. Brannan & W. Boyce, Differential Equations

The components of the net force $F_{\text{net}}$ [in pounds (lb) or Newtons (N)]:

- **Gravitational force** (or **weight of the mass** $W$): it acts downward and is equal to $W = mg$ (where $m$=mass, $g$=acceleration of gravity= 32 ft/s$^2$= 9.8 m/s$^2$).
- **Spring force $F_s$:** the force exerced on the mass by the spring. It is proportional to the total elongation $L + y$ of the spring and opposite to the displacement, i.e.

$$F_s(t) = -k(L + y(t)) \qquad (\textit{Hooke's law})$$

  where $k > 0$ is the **spring constant** (or **stiffness** of the spring).

  At $y = 0$ the mass is in equilibrium. So the spring and the gravitational force balance each others: $mg - kL = 0$.

- **Damping force** (or **resistive force**) $F_d$**:** it may arise from several sources such as air resistance, friction, or additional devices. It is proportional and opposite to the velocity of the mass, i.e.

$$F_d(t) = -\gamma y'(t)$$

  where $\gamma > 0$ is the **damping constant**.

- **External forces or inputs:** Any possible additional force $F(t)$.

Hence: $F_{\text{net}}(t) = W + F_s(t) + F_d(t) + F(t) = -ky(t) - \gamma y'(t) + F(t)$

With $F_{\text{net}}(t) = -ky(t) - \gamma y'(t) + F(t)$, Newton's law of motion: $my''(t) = F_{\text{net}}(t)$ becomes:

$$my''(t) + \gamma y'(t) + ky(t) = F(t)$$

The value of $k$ is obtained from $mg - kL = 0$.

A vibration problem (or oscillator problem) consists in this equation together with two initial conditions:

$$y(0) = y_0 \qquad \text{and} \qquad y'(0) = v_0\,.$$

Four important special cases:

| | |
|---|---|
| Unforced, undamped oscillator: | $my''(t) + ky(t) = 0$ |
| Unforced, damped oscillator: | $my''(t) + \gamma y'(t) + ky(t) = 0$ |
| Forced, undamped oscillator: | $my''(t) + ky(t) = F(t)$ |
| Forced, damped oscillator: | $my''(t) + \gamma y'(t) + ky(t) = F(t)$ |

## Example:

A mass weighting 2 lb stretches a spring 6 in. The mass is pulled down an additional 3 in and at time $t = 0$ it is released at a velocity of 1 ft/s in the upward direction. Assume there is no damping.

Write down the appropriate initial value problem.