Section 4.2: 2nd order linear homogeneous equations

From Section 3.2: every 2nd order DE can be converted into a system of two first order DE's.

In the linear (nonhomogenous) case:

$$y'' + p(t)y' + q(t)y = g(t)$$

the corresponding system is obtained by introducing the state variables:

$$x_1 = y \qquad \text{and} \qquad x_2 = y'$$

We obtain the system of two linear (nonhomogenous) first order DE's:

$$\begin{cases} x_1' = & x_2 \\ x_2' = & -q(t)x_1 - p(t)x_2 + g(t)\,. \end{cases}$$

An initial condition: $y(t_0) = y_0,\, y'(t_0) = y_1$
becomes: $x_1(t_0) = y_0,\, x_2(t_0) = y_1$.

Matrix notation:

$$\mathbf{x}' = \begin{pmatrix} 0 & 1 \\ -q(t) & -p(t) \end{pmatrix} \mathbf{x} + \begin{pmatrix} 0 \\ g(t) \end{pmatrix} \qquad \text{where} \qquad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} y \\ y' \end{pmatrix}$$

with initial condition $\mathbf{x}(t_0) = \mathbf{x}_0 = \begin{pmatrix} y_0 \\ y_1 \end{pmatrix}$.

From Theorem 3.2.1:

Theorem (Theorem 4.2.1)

Consider the second order linear differential equation

$$y'' + p(t)y' + q(t)y = g(t)$$

Suppose the functions p, q and g are continuous on some open interval I. Let t_0 be an element of I.

*Then there **exists** a **unique** solution of the DE satisfying the initial condition $y(t_0) = y_0$ and $y'(t_0) = y_1$, where y_0 and y_1 are any given numbers.*

Example:
Determine the longest interval in which the initial value problem

$$(t^2 - 1)y'' - 3ty' + 4y = \sin(t) \quad \text{with} \qquad y(0) = 2,\ y'(0) = 1$$

have a twice differentiable solution.

Linear operators and 2nd order linear homogenous DEs

Definition

The **operator of differentiation** is the map $D : y \mapsto D[y]$ defined by

$$D[y](t) = \frac{dy}{dt}(t) \qquad \text{for all } t.$$

The **operator of multiplication by the function** p is the operator $p : y \mapsto p[y]$ defined by

$$p[y](t) = p(t)y(t) \qquad \text{for all } t.$$

Both D and p are **linear operators**, that is for all scalars c_1, c_2 and functions y_1, y_2 we have:

$$\begin{aligned} D[c_1 y_1 + c_2 y_2] &= c_1 D y_1 + c_2 D y_2 \\ p[c_1 y_1 + c_2 y_2] &= c_1 p y_1 + c_2 p y_2 \end{aligned}$$

Example: Let y be twice differentiable on the interval I. Then $D^2[y] = D[D[y]]$ is the function with value at $t \in I$ given by $D^2[y](t) = D[D[y]](t) = \frac{d}{dt}\left(\frac{dy}{dt}\right)(t) = \frac{d^2 y}{dt^2}(t)$.

Let p, q two continuous functions on the interval I and set

$$L = D^2 + pD + q = \frac{d^2}{dt^2} + p\frac{d}{dt} + q$$

We can apply L to any function y so that y', y'' exist on I.
If y, y', y'' are continuous on I then

$$L[y] = y'' + py' + q$$

is a continuous function on I.

The value of $L[y]$ at $t \in I$ is

$$L[y](t) = y''(t) + p(t)y'(t) + q(t)\,.$$

The homogeneous linear differential equation $y'' + p(t)y' + q(t)y = 0$ can be rewritten as $L[y] = 0$.

Principle of superposition for linear homogeneous DEs

Theorem (Theorem 4.2.2, Corollary 4.2.3)

$L = D^2 + pD + q$ is a linear operator, i.e. for every twice differentiable function y_1, y_2 on I and every constants c_1, c_2 we have

$$L[c_1 y_1 + c_2 y_2] = c_1 L[y_1] + c_2 L[y_2]$$

If y_1 and y_2 are two solutions of the homogeneous differential equation $L[y] = 0$, so is any linear combination $c_1 y_1 + c_2 y_2$ of y_1 and y_2 (where c_1 and c_2 are arbitrary constants):

$$L[c_1 y_1 + c_2 y_2] = c_1 L[y_1] + c_2 L[y_2] = 0\,.$$

We can extend the notion of linear operators to the case of a homogeneous system of differential equations:

$$\mathbf{x}' = \mathbf{P}(t)\mathbf{x}$$

where the entries of the matrix $\mathbf{P}$ are continuous on an interval I.

The operator $\mathbf{K}$ defined by

$$\mathbf{K}[\mathbf{x}] = \mathbf{x}' - \mathbf{P}(t)\mathbf{x}$$

can be applied to any vector $\mathbf{x}$ for which the components are continuously differentiable on I.

Theorem (Theorem 4.2.4, Corollary 4.2.5)

$\mathbf{K}$ *is a linear operator, i.e. for every continuously differentiable vector functions* $\mathbf{x_1}, \mathbf{x_2}$ *on* I *and constants* c_1, c_2*, we have*

$$\mathbf{K}[c_1\mathbf{x_1} + c_2\mathbf{x_2}] = c_1\mathbf{K}[\mathbf{x_1}] + c_2\mathbf{K}[\mathbf{x_2}]\,.$$

In particular, if $\mathbf{x_1}$ *and* $\mathbf{x_2}$ *are two solutions of the homogeneous differential equations* $\mathbf{K}[\mathbf{x}] = 0$*, so is any linear combination* $c_1\mathbf{x_1} + c_2\mathbf{x_2}$ *of* $\mathbf{x_1}$ *and* $\mathbf{x_2}$*, where* c_1 *and* c_2 *are constants.*

Wronskian and fundamental solutions

Recall from Section 3.3: the Wronskian of two vector functions

$$\mathbf{x_1}(t) = \begin{pmatrix} x_{11}(t) \\ x_{21}(t) \end{pmatrix} \quad \text{and} \quad \mathbf{x_2} = \begin{pmatrix} x_{12}(t) \\ x_{22}(t) \end{pmatrix}$$

on the interval I is the function $W[\mathbf{x_1}, \mathbf{x_2}]$ on I defined by

$$W[\mathbf{x_1}, \mathbf{x_2}](t) = \begin{vmatrix} x_{11}(t) & x_{12}(t) \\ x_{21}(t) & x_{22}(t) \end{vmatrix}.$$

Theorem (Theorem 4.2.6)

Let $\mathbf{x_1}$ and $\mathbf{x_2}$ be two solutions of the homogeneous system of two linear DE $\mathbf{x}' = \mathbf{P}(t)\mathbf{x}$. If the Wronskian $W[\mathbf{x_1}, \mathbf{x_2}]$ is nonzero on the interval I, then $\mathbf{x_1}$ and $\mathbf{x_2}$ form a fundamental set of solutions. The general solution of $\mathbf{x}' = \mathbf{P}(t)\mathbf{x}$ on I is

$$\mathbf{x}(t) = c_1\mathbf{x_1}(t) + c_2\mathbf{x_2}(t)$$

where c_1, c_2 are arbitrary constants.

An initial condition $\mathbf{x}(t_0) = \mathbf{x}_0$ determines the constants c_1 and c_2 uniquely.

We can apply Theorem 4.2.6 to the system

$$\mathbf{x}' = \begin{pmatrix} 0 & 1 \\ -q(t) & -p(t) \end{pmatrix} \mathbf{x}$$

associated with the 2nd order homogenous linear differential equation

$$y'' + p(t)y' + q(t)y = 0$$

Recall the change of variables: $x_1 = y$ and $x_2 = y'$, so that $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$.

The functions y_1 and y_2 are solutions of $y'' + p(t)y' + q(t)y = 0$ if and only if the vector functions $\mathbf{x_1} = \begin{pmatrix} y_1 \\ y_1' \end{pmatrix}$ and $\mathbf{x_2} = \begin{pmatrix} y_2 \\ y_2' \end{pmatrix}$ are solutions of the associated system.

Moreover: $W[\mathbf{x_1}, \mathbf{x_2}](t) = \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix}$.

This motivates the following definition:

Definition

The Wronskian $W[y_1, y_2]$ of the two solutions y_1, y_2 is the function defined for $t \in I$ by

$$W[y_1, y_2](t) = \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix}.$$

Theorem 4.2.6 applied to the system of DE's associated with the 2nd order homogenous linear differential equation

$$y'' + p(t)y' + q(t)y = 0$$

yields the following theorem.

Theorem (Theorem 4.2.7)

Suppose that y_1 and y_2 are two solutions of $y'' + p(t)y' + q(t)y = 0$.
If the Wronskian $W[y_1, y_2]$ of y_1 and y_2 is nonzero on the interval I, then y_1 and y_2 form a fundamental set of solutions. The general solution is given by

$$y(t) = c_1 y_1(t) + c_2 y_2(t)$$

where c_1, c_2 are arbitrary constants.
Two initial conditions $y(t_0) = y_0$ and $y'(t_0) = y_1$ determine the constants c_1, c_2 uniquely.

$$W[y_1, y_2](t) = \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix}.$$

Examples:

- Find the Wronskian of the functions x and xe^x.
- If the Wronskian W of f and g is $3e^{2t}$, and if $f(t) = e^{4t}$, find the function $g(t)$.

How to compute the Wronskian in practice?

Theorem (Theorem 4.2.8, Corollary 4.2.9, Abel Theorem)

The Wronskian W of two solutions of the system $\mathbf{x}' = \mathbf{P}(t)\mathbf{x}$ is given by

$$W(t) = c \exp \int \big(p_{11}(t) + p_{22}(t)\big)\, dt$$

for some constant number c depending on the solutions.
Here: $p_{11}(t) + p_{22}(t) = \mathrm{trace}\mathbf{P}(t)$
The Wronskian of two solutions of the equation $y'' + p(t)y' + q(t)y = 0$ *is given by*

$$W(t) = c \exp \Big(- \int p(t)dt \Big)$$

where c is a constant depending on the solutions.

In particular, the Wronskian is either never zero or always zero in the open interval I.

Method of reduction of order

Consider the equation $y'' + p(t)y' + q(t)y = 0$ of which we know one solution y_1. The method of reduction of order provides a second solution y_2 such that $\{y_1, y_2\}$ is a fundamental system.

- Suppose y_1 is a solution of this equation.
- Put $y_2(t) = v(t)y_1(t)$ and find a condition on v so that y_2 is a solution of the equation.
- Substituting y_2 in the DE equation, one gets:

$$y_1 v'' + (2y_1' + py_1)v' = 0$$

- Letting $w = v'$, we obtain a first-order DE

$$y_1 w' + (2y_1' + py_1)w = 0$$

- Solve and integrate to find v and then y_2.

Example: Consider the differential equation $t^2y'' + 2ty' - 2y = 0$, for $t > 0$. Check that $y_1(t) = t$ is a solution. Use the reduction of order to find a second solution y_2 such that $\{y_1, y_2\}$ is a fundamental set of solutions on $(0, +\infty)$.