EXAMPLES (SECTION 4.2)

- (Ex. 4.2, n° 12) Find the Wronskian of the functions x and xe^x

$$W[x, xe^x] = \begin{vmatrix} x & xe^x \\ 1 & e^x(1+x) \end{vmatrix} = xe^x(1+x) - xe^x = x^2e^x$$

- (Ex. 4.2, n° 18 If $W[f,g](t) = 3e^{2t}$ and $f(t) = e^{4t}$, find $g(t)$

$$3e^{2t} = \begin{vmatrix} e^{4t} & g(t) \\ 4e^{4t} & g'(t) \end{vmatrix} = e^{4t}(g'(t) - 4g(t))$$

i.e. $g'(t) - 4g(t) = 3e^{-2t}$ (1st order linear DE in g)

Integrating factor $\mu(t) = e^{-4t}$ because $\int -4dt = -4t + C$

$$(e^{-4t}g(t))' = 3e^{-2t}e^{-4t} = 3e^{-6t}$$

$$e^{-4t}g(t) = \int 3e^{-6t}dt = \frac{1}{2}e^{-6t} + C$$

$(te^{-t})' = e^{-t} - te^{-t}$
$= e^{-t}(1-t) = -t$

So: $g(t) = \frac{1}{2}e^{-2t} + Ce^{4t}$, $C \in \mathbb{R}$

- (Ex. 4.2, n° 30) $t^2y'' + 2ty' - 2y = 0$ for $t > 0$. Check that $y_1(t) = t$ is a solution. Use the reduction of order to find a second solution y_2 s.t. $\{y_1, y_2\}$ is a fundamental system of solutions on $(0, +\infty)$

(1) $y_1(t) = t$ is a solution because $y_1' = 1$, $y_1'' = 0$ and hence $t^2y_1'' + 2ty_1 - 2y_1$
$= 2t - 2t = 0$

(2) Set $y_2(t) = v(t)y_1(t) = v(t).t$. Substitute into the DE:

$$y_2 = tv, \quad y_2' = v + tv', \quad y_2'' = v' + tv'' + v' = tv'' + 2v'$$

$$0 = t^2y_2'' + 2ty_2' - 2y_2 = t^2(tv'' + 2v') + 2t(\cancel{v} + tv') - \cancel{2}tv = t^3v'' + 4t^2v'$$

Division by $t^2 > 0$ yields: $tv'' + 4v' = 0$. Substitute $w = v'$:

$tw' + 4w = 0$ i.e. $w' + \frac{4}{t}w = 0$ i.e. $\frac{1}{w}w' = -\frac{4}{t}$

$$\int \frac{dw}{w} = -4\int \frac{dt}{t}, \text{ i.e. } \ln|w| = \underbrace{-4\ln|t|}_{\ln\left(\frac{1}{t^4}\right)} + C_1, \text{ i.e. } v' = w = C_2\frac{1}{t^4}$$

Thus $v = \int v'dt = C_2\int \frac{dt}{t^4} = C_3\frac{1}{t^3} + C_4$ (where $C_3 = -\frac{C_2}{3}$)

<u>Conclusion</u>: $y_2 = tv = C_3\frac{1}{t^2} + C_4t$. Since $y_1 = t$, can choose f.i. $C_4 = 0$, any $C_3 \neq 0$.