4.3: Linear homogeneous equations with constant coefficients

Consider a homogeneous second order linear differential equation

$$ay'' + by' + cy = 0$$

where a, b and c are given real numbers (with $a \neq 0$).

Letting $x_1 = y$ and $x_2 = y'$, this equation transforms into the system

$$\mathbf{x}' = \mathbf{A}\mathbf{x} = \begin{pmatrix} 0 & 1 \\ -c/a & -b/a \end{pmatrix} \mathbf{x} \qquad \text{where} \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} y \\ y' \end{pmatrix}$$

Definition

The **characteristic equation** (or **auxiliary equation**) of $ay'' + by' + cy = 0$ is

$$a\lambda^2 + b\lambda + c = 0\,.$$

The **characteristic polynomial** of $ay'' + by' + cy = 0$ is $\quad a\lambda^2 + b\lambda + c$.

The characteristic equation of $ay'' + by' + cy = 0$ is the characteristic equation of $\mathbf{A}$. Its solutions (also called roots) are the eigenvalues of $\mathbf{A}$.

If λ is an eigenvalue of $\mathbf{A}$ then one can check that $v = \begin{pmatrix} 1 \\ \lambda \end{pmatrix}$ is an eigenvector of $\mathbf{A}$ for the eigenvalue λ.

Let λ be a root of the characteristic equation $a\lambda^2 + b\lambda + c = 0$. The following theorem specifies how to associate solutions to λ.

Theorem (Theorem 4.3.1)

- *If λ is a root of characteristic equation, then the function $y(t) = e^{\lambda t}$ is a solution of the equation $ay'' + by' + cy = 0$.*
- *If λ is a root of characteristic equation, then the vector function $\mathbf{x}(t) = \begin{pmatrix} e^{\lambda t} \\ \lambda e^{\lambda t} \end{pmatrix}$ is a solution of $\mathbf{x}' = \mathbf{A}\mathbf{x}$.*

Remark:

- The solution of $ay'' + by' + cy = 0$ is the first component of the solution of $\mathbf{x}' = \mathbf{A}\mathbf{x}$.
- We may substitute $y = e^{\lambda t}$ into $ay'' + by' + cy = 0$ and find directly that $y = e^{\lambda t}$ is a solution if and only if λ satisfies the characteristic equation.

Theorem (Theorem 4.3.2)

Let λ_1 *and* λ_2 *be the (possibly equal) roots of the characteristic equation*

$$a\lambda^2 + b\lambda + c = 0\,.$$

Then the general solution of the differential equation

$$ay'' + by' + cy = 0$$

is:

- $y(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$ *if* $\lambda_1 \neq \lambda_2$ *are real,*
- $y(t) = c_1 e^{\lambda_1 t} + c_2 t e^{\lambda_1 t}$ *if* $\lambda_1 = \lambda_2$*,*
- $y(t) = c_1 e^{\mu t} \cos(\nu t) + c_2 e^{\mu t} \sin(\nu t)$ *if* $\overline{\lambda_2} = \lambda_1 = \mu + i\nu$*.*

In the above formulas, c_1 *and* c_2 *are arbitrary constants.*

Example:

Determine the general solution of $y'' + 5y' + 6y = 0$

Theorem (Theorem 4.3.2, continued)

The general solution of the corresponding system of linear differential equations

$$\mathbf{x}' = \mathbf{A}\mathbf{x} = \begin{pmatrix} 0 & 1 \\ -c/a & -b/a \end{pmatrix} \mathbf{x}$$

is:

- $\mathbf{x}(t) = c_1 e^{\lambda_1 t} \begin{pmatrix} 1 \\ \lambda_1 \end{pmatrix} + c_2 e^{\lambda_2 t} \begin{pmatrix} 1 \\ \lambda_2 \end{pmatrix}$ *if* $\lambda_1 \neq \lambda_2$ *are real,*
- $\mathbf{x}(t) = c_1 e^{\lambda_1 t} \begin{pmatrix} 1 \\ \lambda_1 \end{pmatrix} + c_2 e^{\lambda_1 t} \begin{pmatrix} t \\ 1 + \lambda_1 t \end{pmatrix}$ *if* $\lambda_1 = \lambda_2$,
- $\mathbf{x}(t) = c_1 e^{\mu t} \begin{pmatrix} \cos(\nu t) \\ \mu \cos(\nu t) - \nu \sin(\nu t) \end{pmatrix} + c_2 e^{\mu t} \begin{pmatrix} \sin(\nu t) \\ \mu \sin(\nu t) + \nu \cos(\nu t) \end{pmatrix}$

  if $\overline{\lambda_2} = \lambda_1 = \mu + i\nu$.

In the above formulas, c_1 *and* c_2 *are arbitrary constants.*

Example:

Determine the system of linear equations corresponding to $y'' + 5y' + 6y = 0$ and its general solution.

Phase portraits

Goal: to study the dynamical system $\mathbf{x}' = \mathbf{A}\mathbf{x} = \begin{pmatrix} 0 & 1 \\ -c/a & -b/a \end{pmatrix} \mathbf{x}$

associated with the homogeneous second order linear differential equation

$$ay'' + by' + cy = 0$$

The state variables are $x_1 = y$ and $x_2 = y'$ and $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} y \\ y' \end{pmatrix}$.

We apply the results of Sections 3.3, 3.4, 3.5 to dynamical systems with

$$\mathbf{A} = \begin{pmatrix} 0 & 1 \\ -c/a & -b/a \end{pmatrix}$$

Recall: the equilibrium solutions (or critical points) of $\mathbf{x}' = \mathbf{A}\mathbf{x}$ are the solutions of $\mathbf{A}\mathbf{x} = 0$ (i.e. the constant solutions $\mathbf{x} = \mathbf{constant}$).

Since $\det \mathbf{A} = c/a$, we obtain:

- $\mathbf{x}_{\mathrm{eq}} = \mathbf{0} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is the unique equilibrium solution if and only if $c \neq 0$.
- If $c = 0$, there is one line of equilibrium solutions $\mathbf{x}_{\mathrm{eq}}$.

If $\mathbf{x}_{\mathrm{eq}} = \mathbf{0}$ is unique, then its type (nodal source/nodal sink/saddle/spiral sink/spiral source/center/stable or unstable proper node) and its stability (stable/unstable/asymptotically stable) are determined as in Tables 3.3.1, 3.4.1, 3.5.1.

Example:
Consider the 2nd order homogenous linear differential equations:

$$y'' + y' - 6y = 0$$

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- From the roots of of the characteristic equation, determine the type each critical point of the corresponding dynamical system.
- Use the general solution to find a two-parameter family of trajectories of the corresponing dynamical system.
- Sketch the phase portrait, including straight line orbits, from this family of trajectories.