Section 4.4: Vibrations. Harmonic oscillators

Main Topics:

- **Mechanical vibrations (systems spring-mass)**
- **Harmonic oscillators**
- **Examples of second order differential equations.**

Recall the equation describing the dynamics (or the vibrations) of a spring-mass system:

$$my'' + \gamma y' + ky = F$$

where

- the unknown function $y = y(t)$ describes the motion of the mass at time t,
- m is the **mass**,
- k the **spring constant**,
- γ the **damping factor**,
- $F = F(t)$ an **external force** applied on the system.

(m, k and γ are positive constants).

The initial conditions:

$$y(0) = y_0 \qquad \text{and} \qquad y'(0) = v_0$$

(specifying the initial position y_0 and the initial velocity v_0)

uniquely determine the motion.

- If $F = 0$ we say that the system is **unforced** or **free**.
- Free spring-mass systems are known as **harmonic oscillators**.

Undamped free vibrations

This case corresponds to $\gamma = 0$ and $F = 0$.

The equation of the motion is

$$my'' + ky = 0$$

Setting

$$\omega_0^2 = k/m$$

this equation becomes

$$y'' + \omega_0^2 y = 0\,.$$

Characteristic equation: $\lambda^2 + \omega_0^2 = 0$, with roots $\lambda = \pm i\omega_0$.

The general solution is:

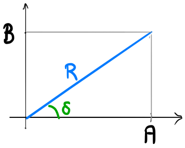
$$y(t) = A\cos(\omega_0 t) + B\sin(\omega_0 t)$$

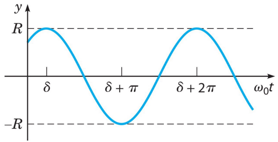
for arbitrary constants A and B.

The **phase-amplitude form** of the general solution $y(t) = A\cos(\omega_0 t) + B\sin(\omega_0 t)$ is

$$y(t) = R\cos(\omega_0 t - \delta)$$

with $\begin{cases} R = \sqrt{A^2 + B^2} \\ A = R\cos\delta \\ B = R\sin\delta \end{cases}$





Simple harmonic motion $y = R\cos(\omega_0 t - \delta)$.

Figure 4.4.3 in J. Brannan & W. Joyce

so that $\begin{cases} \cos\delta = \frac{A}{\sqrt{A^2+B^2}} \\ \sin\delta = \frac{B}{\sqrt{A^2+B^2}}. \end{cases}$

The graph of $y(t) = R\cos(\omega_0 t - \delta)$ is a displaced cosine function.
This is a **periodic motion** (or **simple harmonic motion**) of the mass:

- $T = \frac{2\pi}{\omega_0}$ is the **period**,
- $\omega_0 = \sqrt{k/m}$ is the **natural frequency**,
- δ is the **phase**,
- R is the **amplitude**.

Example: Determine ω_0, R and δ such that $y = -\cos t + \sqrt{3}\sin t$ can be written as $y = R\cos(\omega_0 t - \delta)$.

Damped free vibrations

This case corresponds to $\gamma \neq 0$ and $F = 0$.

The equation of the motion is

$$my'' + \gamma y' + ky = 0$$

Characteristic equation: $m\lambda^2 + \gamma\lambda + k = 0$ with roots:

$$\lambda_1, \lambda_2 = \frac{-\gamma \pm \sqrt{\gamma^2 - 4km}}{2m} = \frac{\gamma}{2m}\Big(-1 \pm \sqrt{1 - \frac{4km}{\gamma^2}}\Big)$$

The solutions of the equation of the motion are of three different types, depending on the sign of $\gamma^2 - 4km$.

$$\lambda_1, \lambda_2 = \frac{-\gamma \pm \sqrt{\gamma^2 - 4km}}{2m} = \frac{\gamma}{2m}\Big(-1 \pm \sqrt{1 - \frac{4km}{\gamma^2}}\Big)$$

- **Underdamped harmonic motion:** $\gamma^2 - 4km < 0$

  Complex conjugate roots $\lambda_1 = \mu + i\nu$, $\lambda_2 = \overline{\lambda_1}$.

  General solution: $y(t) = e^{-\gamma t/2m}(A\cos(\nu t) + B\sin(\nu t))$

  with $\mu = -\gamma/2m < 0$ and $\nu = \dfrac{\sqrt{4km - \gamma^2}}{2m} > 0$.

- **Critically damped harmonic motion:** $\gamma^2 - 4km = 0$

  Repeated *negative* roots $\lambda_1 = \lambda_2 = -\gamma/2m < 0$.

  General solution: $y(t) = (A + Bt)e^{-\gamma t/2m}$

- **Overdamped harmonic motion:** $\gamma^2 - 4km > 0$

  Two distinct *negative* eigenvalues λ_1, λ_2 (because $\sqrt{\gamma^2 - 4km} < \gamma$).

  General solution: $y(t) = Ae^{\lambda_1 t} + Be^{\lambda_2 t}$

Remark: In all cases $\lim_{t\to+\infty} y(t) = 0$
(because of the damping, as t increases, the motion decreases and eventually stops)

A more detailed study of the underdamped harmonic motion: $\gamma^2 - 4km < 0$

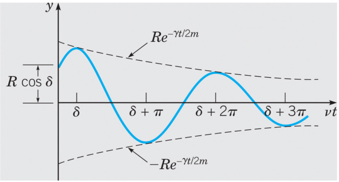
General solution: $y(t) = e^{-\gamma t/2m}(A\cos(\nu t) + B\sin(\nu t))$

with $-\gamma/2m < 0$ and

$$\nu = \frac{\sqrt{4km - \gamma^2}}{2m} = \frac{\gamma}{2m}\sqrt{1 - \frac{4km}{\gamma^2}} > 0$$

As for the simple harmonic motion, set:

$$\begin{cases} R = \sqrt{A^2 + B^2} \\ A = R\cos\delta \\ B = R\sin\delta \end{cases}$$



Damped vibration; $y = Re^{-\gamma t/2m}\cos(\nu t - \delta)$.

Figure 4.4.5 in J.Brannan & W. Joyce

and write: $y(t) = Re^{-\gamma t/2m}\cos(\nu t - \delta)$

- the motion oscillates between the curves $y = Re^{-\gamma t/2m}$ (damped oscillation),
- ν is the **quasi-frequency**,
- $T_d = \frac{2\pi}{\nu}$ is the **quasi-period**.
- As $\gamma \to 2\sqrt{km}$ we have $\nu \to 0$ and $T_d \to \infty$ and there is no oscillation (critically damped motion). The value $\gamma = 2\sqrt{km}$ is called the **critical damping**.

Phase portraits for harmonic oscillators

Phase portraits are obtained from corresponding systems of first order differential equations:

$$\mathbf{x}' = \mathbf{A}\mathbf{x} = \begin{pmatrix} 0 & 1 \\ -k/m & -\gamma/m \end{pmatrix} \mathbf{x}$$

The origin is the unique equilibrium point because A is invertible
(as $\det(A) = k/m \neq 0$).

The roots of the characteristic equation $m\lambda^2 + \gamma\lambda + k = 0$ are the eigenvalues of $\mathbf{A}$.

They give the nature of the equilibrium point of the different harmonic oscillators:

- a **stable center** for the undamped harmonic oscillator,
- a **spiral sink** for the underdamped harmonic oscillator,
- a **nodal sink** for both the overdamped and the critically damped harmonic oscillators.

Example:

A mass of weight 32 pounds is attached at the bottom end of a spring of natural lenght 6 ft. At the equilibrium, the spring measures 9.2 ft. The mass is initially released from rest at a point of 2 feet above the equilibrium position. Assume that the surrounding medium offers a resistance which is twice the instantaneous velocity.

- Write an initial value problem describing for the displacement $y = y(t)$ of the mass as a function of the time t.
- Solve the initial value problem and determine $y(t)$.
- Classify the type of harmonic oscillator. What are the parameters characterizing this motion?
- Describe the motion of the mass for very large values of t. Sketch the function $y(t)$.
- Draw a phase portrait of the equivalent dynamical system. Give the type of equilibrium point.
- Describe the trajectory of the given initial value problem.