4.4: Nonhomogenous equations. Method of undetermined coefficients

Main Topics:

Second order nonhomogeneous linear differential equations

$$y'' + q(t)y' + p(t)y = g(t)$$

Nonhomogeneous: $g(t) \neq 0$.

- **Form of the general solution**
- **Method of undetermined coefficients to find a particular solution**

Recall the linear operator $L = D^2 + pD + q = \frac{d^2}{dt^2} + p\frac{d}{dt} + q$
So $L[y] = y'' + p(t)y' + q(t)$.

The linear differential equation $y'' + q(t)y' + p(t)y = g(t)$
can be written in terms of L as $L[y] = g(t)$.

Definition

The homogeneous linear equation (with same p and q and $g = 0$)

$$y'' + q(t)y' + p(t)y = 0 \qquad \text{i.e.} \quad L[y] = 0\,,$$

is called the **homogeneous equation corresponding to** $L[y] = g(t)$.

Suppose Y_1 and Y_2 are solutions of $L[y] = g(t)$, i.e.

$$L[Y_1] = g(t) \qquad \text{and} \qquad L[Y_2] = g(t)$$

Then $L[Y_1 - Y_2] = L[Y_1] - L[Y_2] = g(t) - g(t) = 0$, that is
$Y_1 - Y_2$ is a solution of the corresponding homogeneous equation $L[y] = 0$.

Theorem (Theorem 4.5.1)

Suppose

- Y_1 *and* Y_2 *are solutions of* $L[y] = g(t)$
- y_1 *and* y_2 *form a fundamental set of solutions of* $L[y] = 0$.

Then: $Y_1 - Y_2 = c_1 y_1 + c_2 y_2$ *for some constants* c_1, c_2.

Theorem (Theorem 4.5.2)

The general solution of the nonhomogeneous linear differential equation $L[y] = g(t)$ *is of the form*

$$y(t) = c_1 y_1(t) + c_2 y_2(t) + Y(t)$$

where

- c_1, c_2 *are arbitrary constants,*
- Y *is a particular solution of* $L[y] = g(t)$.

In other words:

Theorem (Theorem 4.5.2, restated)

The general solution of the nonhomogeneous linear differential equation $L[y] = g(t)$ *is of the form*

$$y(t) = y_c(t) + Y(t)$$

where

- y_c *is the general solution of the corresponding homogenous differential equation* $L[y] = 0$ *(y_c is often called the **complementary solution**)*
- Y *is a particular solution of* $L[y] = g(t)$.

General strategy to solve the nonhomogeneous linear differential equation $L[y] = g(t)$:

- find the general solution y_c of the corresponding homogenous equation $L[y] = 0$
- find **one** particular solution Y of the nonhomogenous equation $L[y] = g$
- add them together.

We know how to determine y_c when $L[y] = 0$ has constant coefficients (Section 4.3)
(In general, i.e. if $p(t)$ and $q(t)$ are not constant functions, finding y_c is a more difficult issue)

We now look at how **to find a particular solution** Y of $L[y] = g$.
Two methods:

- Method of undetermined coefficients (here, in Section 4.5. It applies to DE with constant coefficients)
- Method of variation of parameters (Section 4.7)

Method of undetermined coefficients

Apply this method to a **constant coefficient** differential equation of the form

$$ay'' + by' + c = g(t)$$

where b, c are constants and $g(t)$ is of a specific form given below.

Method:

- find the general solution y_c of the homogeneous equation $ay'' + by' + c = 0$
- make **a guess** on the form of the expected particular function Y.
  The choice of Y depends on:
  - ◇ the form of $g(t)$
  - ◇ the form of the two fundamental solutions of $ay'' + by' + c = 0$.

  The guess for Y will depend on a finite number of parameters $A, B, \ldots$.
- Substitute the guess for Y in the differential equation $ay'' + by' + c = g(t)$ and determine the parameters.

Example: If $g(t) = e^{\alpha t}$, choose $Y(t) = At^s e^{\alpha t}$.
Apply this "guess" to solve the following differentials equations:

- $y'' - 3y' - 4y = 4e^{3t}$
- $y'' - 3y' - 4y = e^{-t}$