Section 4.6: Forced vibrations, frequency response, and resonance

Main Topics:

- **Forced vibrations with damping**
- **Forced vibrations without damping**
- **Frequency and resonance.**

Recall the equation describing the dynamics (or the vibrations) of a spring-mass system:

$$my'' + \gamma y' + ky = F$$

where

- the unknown function $y = y(t)$ describes the motion of the mass at time t,
- m is the **mass**,
- k the **spring constant**,
- γ the **damping factor**,
- $F = F(t)$ an **external force** applied on the system.

(m, k and γ are positive constants).

Dividing all terms by m we get:

$$y'' + 2\delta y' + \omega_0^2 y = f$$

where

- $\omega_0 = \sqrt{k/m}$ is the **natural frequency**,
- $\delta = \frac{\gamma}{2m}$ is the **phase**,
- $f(t) = \frac{F(t)}{m}$.

Suppose that the external force is *periodic*, that is that:

$$f(t) = A\cos(\omega t) \quad \text{or} \quad f(t) = A\sin(\omega t)$$

(or a linear combination of both).

For an input force equal to $A\cos(\omega t)$ (or equal to $A\sin(\omega t)$)

- A is the **amplitude** of the input force
- ω is the **frequency**.

It is convenient to consider both terms simultaneously via complex exponentials:

$$f(t) = A\cos(\omega t) + iA\sin(\omega t) = Ae^{i\omega t}$$

Remark: the magnitude of force must be a real function.
Complex numbers are a *tool* to deal with both sin and cos terms at once.

The results for $f(t) = A\cos(\omega t)$ will be obtained from those for $f(t) = Ae^{i\omega t}$ by taking real parts; those for $f(t) = A\sin(\omega t)$ by taking imaginary parts.

Forced vibrations with damping ($\gamma = 2m\delta \neq 0$)

The general solution of the differential equation

$$y'' + 2\delta y' + \omega_0^2 y = Ae^{i\omega t}$$

is of the form $\quad y(t) = y_c(t) + Y(t) \quad$ where

- y_c is the general solution of $y'' + 2\delta y' + \omega_0^2 y = 0$,
- $Y(t)$ is a particular solution of the nonhomogenous equation.

- y_c is computed as for the damped free vibrations (Section 4.4).

A fundamental system of solutions y_1, y_2 is obtained from the roots λ_1, λ_1 of the characteristic equation

$$\lambda^2 + 2\delta\lambda + \omega_0^2 = 0.$$

The nature of a fundamental system of solutions y_1, y_2 depends on the sign of the discriminant $\delta^2 - \omega_0^2$.

Remark: Since $\omega_0 = \sqrt{k/m}$ and $\delta = \gamma/2m$, we have

$$\delta^2 - \omega_0^2 = \frac{\gamma^2}{4m^2} - \frac{k}{m} = \frac{1}{m^2}(\gamma^2 - 4km)$$

Hence:
the signs ($> 0, = 0, < 0$) of $\delta^2 - \omega_0^2$ (here) and $\gamma^2 - 4km$ (section 4.4) are the same (as they should: we just divided the initial equation by m).

- A particular solution $Y(t)$ of the differential equation $y'' + 2\delta y' + \omega_0^2 y = Ae^{i\omega t}$ can be determined by the method of undetermined constants:

$Y(t) = Ce^{i\omega t}$ with C constant determined by substitution into the DE.

$$\left((i\omega)^2 + 2\delta(i\omega) + \omega_0^2\right) Ce^{i\omega t} = Ae^{i\omega t}$$

i.e.

$$C = \frac{A}{(i\omega)^2 + 2\delta(i\omega) + \omega_0^2}$$

Therefore a particular (complex) solution of the differential equation is

$$Y(t) = \frac{A}{(i\omega)^2 + 2\delta(i\omega) + \omega_0^2}\; e^{i\omega t}$$

Remark: Above we considered $f(t) = Ae^{i\omega t}$.

If $f(t) = A\cos(\omega t)$, then a particular solution is $Y_{\mathrm{Re}}(t) = \mathrm{Re}\, Y(t)$:

$$Y_{\mathrm{Re}}(t) = A\frac{(\omega_0^2 - \omega^2)\cos(\omega t) + 2\delta\omega\sin(\omega t)}{(\omega_0^2 - \omega^2)^2 + 4\delta^2\omega^2}$$

This solution is a steady oscillation with the *same frequency* ω as the external force.

- The function Y_{Re} is referred to as the **steady-state solution** or **steady-state response** or **forced response**.

Suppose $f(t) = A\cos(\omega t)$**:**

The general solution solution of the differential equation $y'' + 2\delta y' + \omega_0^2 y = f(t)$ is

$$y(t) = y_c(t) + Y_{\mathrm{Re}}(t)$$

The explicit form of y_c (already determined in Section 4.4) depends on the nature of the roots of the characteristic equation $\lambda^2 + 2\delta\lambda + \omega_0^2 = 0$.

The possible cases can be *rewritten* in terms of δ (phase) and ω_0 (natural frequency), as follows.

The roots of the characteristic equation are:

- $-\delta \pm \sqrt{\delta^2 - \omega_0^2}$ if $\delta^2 - \omega_0^2 > 0$
- $-\delta$ (repeated) if $\delta^2 - \omega_0^2 = 0$
- $-\delta \pm i\sqrt{\omega_0^2 - \delta^2}$ if $\delta^2 - \omega_0^2 < 0$

As noticed in Section 4.4: $\lim_{t\to+\infty} y_c(t) = 0$.

In all three cases, $y_c = e^{-\delta t} \times$ (function of t) and one can check that $\lim_{t\to+\infty} y_c(t) = 0$.

- The function y_c is referred to as the **transient solution**.

However, the solution $y(t) = y_c(t) + Y_{\mathrm{Re}}(t)$ generally **does not** die out as t increases.

Example:

A mass m of 0.2 kg is attached to a spring of spring constant $k = 2\ N/m$. The mass is initially released from rest at 0.5 m below the equilibrium point. An external periodic force $F(t) = 5\cos(4t)$ with period $T = \pi/2$ sec is applied to the mass. Moreover, the medium offers a resistance corresponding to a damping constant of value $\gamma = 1.2 Nsec/m$.

- Write the IVP modeling the displacement y of the mass as a function of time.
- What are the phase δ and the natural frequence ω_0?
- Determine a (complex-value) particular solution $Y(t)$ and the corresponding steady-state solution $Y_{\mathrm{Re}}(t)$.
- Determine $y(t)$.

Frequency response

The **frequency response** of the system with forcing function $f(t) = A\cos(\omega t)$ [or $f(t) = A\sin(\omega t)$] is the ratio

$$G(i\omega) = \frac{Y(t)}{Ae^{i\omega t}} = \frac{1}{(i\omega)^2 + 2\delta(i\omega) + \omega_0^2}$$

It does not depend on the time.

In the trigonometric form: $G(i\omega) = \mid G(i\omega) \mid e^{-i\phi(\omega)}$
where:

- the modulus of $G(i\omega)$ is the **gain** of the frequency response:

$$\mid G(i\omega) \mid = \frac{1}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\delta^2\omega^2}}$$

- the argument of $G(i\omega)$ is the following angle, called the **phase of the frequency response**

$$\phi(\omega) = \arccos\left(\frac{\omega_0^2 - \omega^2}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\delta^2\omega^2}}\right)$$

The (complex) particular solution can be written as

$$Y(t) = Ae^{i\omega t}G(i\omega) = A \mid G(i\omega) \mid e^{i(\omega t - \phi(\omega))}$$

The particular solution for the **harmonic input** $f(t) = A\cos(\omega t)$ can be written:

$$Y_{\text{Re}}(t) = A \mid G(i\omega) \mid \cos(\omega t - \phi(\omega))$$

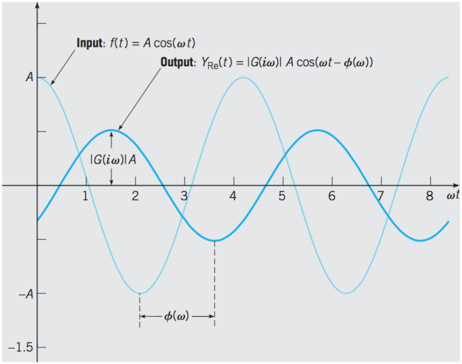

FIGURE 4.6.5 The steady-state response $Y_{\text{Re}} = |G(i\omega)|A\cos(\omega t - \phi(\omega))$ of a spring-mass system due to the harmonic input $f(t) = A\cos\omega t$.

The amplitude of the output $Y_{\text{Re}}(t)$ is $A \mid G(i\omega) \mid$
The phase of the output $Y_{\text{Re}}(t)$ has a phase shift of magnitude $\phi(\omega)$

How do the gain and the phase of the frequency response depend on the frequency ω of the harmonic input?

The explicit expressions are complicated:

$$| G(i\omega) |= \frac{1}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\delta^2\omega^2}}$$

$$\phi(\omega) = \arccos\left(\frac{\omega_0^2 - \omega^2}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\delta^2\omega^2}}\right).$$

- For low frequency inputs: $\lim_{\omega \to 0^+} |G(i\omega)| = 1/\omega_0^2 = m/k$
- For very high frequency inputs: $\lim_{\omega \to +\infty} |G(i\omega)| = 0$
- Maximal gain at $\omega = \omega_{\max}$ for which $\frac{d}{d\omega}|G(i\omega)| = 0$:

$$\omega_{\max}^2 = \omega_0^2 - 2\delta = \omega_0^2(1 - \frac{\gamma^2}{2mk})$$

$$|G(i\omega_{\max})| = \frac{m}{\gamma\omega_0\sqrt{1 - (\gamma^2/4mk)}}$$

Remark: $0 < \omega_{\max} < \omega_0$ and $\omega_{\max} \approx \omega_0$ if the damping $\gamma \approx 0$.

Resonances

- For $\gamma \approx 0+$ the gain is very large and tends to $+\infty$:

$$|G(i\omega_{\max})| = \frac{m}{\gamma\omega_0\sqrt{1-(\gamma^2/4mk)}} \approx \frac{m}{\gamma\omega_0}$$

- The maximal amplitude of the output equals the product $A \mid G(i\omega_{\max}) \mid$.
- When the steady-state response oscillates with a much greater amplitude than the input, the system is said to be in **resonance**.

  The frequency at which the maximal amplitude of the steady-state response occur is called the **resonant frequency** of the system.
- Resonance must be taken into account:

  - when designing a system which vibrates or oscillates, it is important to keep track of its resonance properties. Miscalculations can lead to catastrophs!

  - to build instruments as seismographs (intended to detect peaks of signals) use resonance properties.

Example: [Example 1 in Section 4.6, p. 263]
Consider the IVP

$$y'' + 0.25y' + 2y = 2\cos(\omega t)$$

with $y(0) = 2$, $y'(0) = 0$.
One can compute that the IVP has the following complex-value particular solution

$$Y(t) = \frac{3}{(i\omega)^2 + i\omega/8 + 1} e^{i\omega t}$$

- Determine the steady-state part of the solution.
- Find the gain function $\mid G(i\omega) \mid$ of the system.
- Find the maximum value of $\mid G(i\omega) \mid$ and the frequency ω for which it occurs.

Forced vibrations without damping

In this case $\gamma = 0$ and $\delta = 0$. The differential equation for an undamped forced oscillator is

$$y'' + \omega_0^2 y = f(t)$$

We suppose as before that $f(t) = A\cos(\omega t)$.

The **general solution** is $y = y_c + Y_p$ where:

- $y(t) = c_1 \cos(\omega_0 t) + c_2 \sin(\omega_0 t)$, with c_1, c_2 arbitrary constants, is the general solution of the homogenous equation $y'' + \omega_0^2 y = 0$
  [characteristic equation: $\lambda^2 + \omega_0^2 = 0$; roots $\lambda = \pm i\omega_0$].
- Y_p is a particular solution of the given DE.

The **particular solution** Y_p is computed with the method of undetermined constants. Its form depends on ω and ω_0.

- If $\omega \neq \omega_0$ then $e^{i\omega t}$ is not a solution of homogeneous equation.
  We substitute $Y(t) = Ce^{i\omega t}$ in the DE to determine C. Then $Y_p(t) = \mathrm{Re}\, Y(t)$.
  Computation gives $Y_p(t) = \dfrac{A}{(\omega_0^2 - \omega^2)} \cos(\omega t)$.
- If $\omega = \omega_0$ then $e^{i\omega_0 t}$ is a solution of homogeneous equation but $te^{i\omega_0 t}$ is not.
  We substitute $Y(t) = Cte^{i\omega_0 t}$ in the DE to determine C. Then $Y_p(t) = \mathrm{Re}\, Y(t)$.
  Computation gives $Y_p(t) = \dfrac{A}{2\omega_0} t \sin(\omega_0 t)$.