Section 4.7: Variations of parameters

The **method of variation of parameters**

(also called **method of variation of constants** or **method of Lagrange**)

is a method for **finding a particular solution** of:

- systems of first-order linear differential equations $\mathbf{x}' = \mathbf{P}(t)\mathbf{x} + \mathbf{g}(t)$
- second order nonhomogeneous linear differential equations $y'' + p(t)y' + q(t)y = g(t)$

Unlike the method of undetermined constants:

- we do not assume constant coefficients,
- we do not assume that $g(t)$ has a special form.

But: it is more difficult to apply it.

Consider the system of linear differential equations $\mathbf{x}' = \mathbf{P}(t)\mathbf{x} + \mathbf{g}(t)$ where

$$\mathbf{P}(t) = \begin{pmatrix} p_{11}(t) & p_{12}(t) \\ p_{21}(t) & p_{22}(t) \end{pmatrix} \quad \text{and} \quad \mathbf{g}(t) = \begin{pmatrix} g_1(t) \\ g_2(t) \end{pmatrix}.$$

Let $\mathbf{x}_1$ and $\mathbf{x}_2$ be solutions of the homogenous system $\mathbf{x}' = \mathbf{P}(t)\mathbf{x}$ on an open interval I.

If $\mathbf{x}_1(t) = \begin{pmatrix} x_{11}(t) \\ x_{21}(t) \end{pmatrix}$ and $\mathbf{x}_2(t) = \begin{pmatrix} x_{12}(t) \\ x_{22}(t) \end{pmatrix}$, then set $\mathbf{X}(t) = \begin{pmatrix} x_{11}(t) & x_{12}(t) \\ x_{21}(t) & x_{22}(t) \end{pmatrix}$.

Theorem (Theorem 4.7.1)

Suppose that all entries of $\mathbf{P}$ *and of* $\mathbf{g}$ *are continuous on the interval* I*.*

Let $\mathbf{x}_1$ *and* $\mathbf{x}_2$ *be a fundamental set of solutions of* $\mathbf{x}' = \mathbf{P}(t)\mathbf{x}$*.*
Then:

- $\mathbf{X}(t)$ *is invertible for all* $t \in I$*.*
- *A particular solution* $\mathbf{x}_p$ *of* $\mathbf{x}' = \mathbf{P}(t)\mathbf{x} + \mathbf{g}(t)$ *on* I *is*

$$\mathbf{x}_p(t) = \mathbf{X}(t) \int \mathbf{X}^{-1}(t)\mathbf{g}(t)\, dt$$

- *The general solution of* $\mathbf{x}' = \mathbf{P}(t)\mathbf{x} + \mathbf{g}(t)$ *on* I *is of the form:*

$$\mathbf{x}(t) = c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t) + \mathbf{x}_p(t),$$

  where c_1, c_2 *are arbitrary constants.*

Example:

Find the general solution of the system

$$\mathbf{x}' = \begin{pmatrix} 1 & -1 \\ 2 & 4 \end{pmatrix} \mathbf{x} + \begin{pmatrix} t \\ 2t \end{pmatrix}$$

Remarks:

- Recall that $\mathbf{X}^{-1}(t) = \dfrac{1}{W[\mathbf{x}_1, \mathbf{x}_2](t)} \begin{pmatrix} x_{22}(t) & -x_{12}(t) \\ -x_{21}(t) & x_{11}(t) \end{pmatrix}$
- Origin of the name "variation of parameters" or "variation of constants": the expression for $\mathbf{x}_p$ is determined by substituting the vector-function

$$\mathbf{x}_p(t) = u_1(t)\mathbf{x}_1(t) + u_2(t)\mathbf{x}_2(t) = \mathbf{X}(t) \begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix}$$

  in the system of differential equations.
  That is: in the general solution

$$c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t)$$

  of the corresponding homogenous system $\mathbf{x}' = \mathbf{P}\mathbf{x}$, we have replaced the constants c_1 and c_2 with the functions $u_1(t)$ and $u_2(t)$.

  The substitution yields

$$\begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix} = \int \mathbf{X}^{-1}(t)\mathbf{g}(t)\, dt$$

- Main difficulties of the method:
  - ◇ evaluating the antiderivative $\int \mathbf{X}^{-1}(t)\mathbf{g}(t)\, dt$
  - ◇ finding the fundamental solutions $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$ for systems with nonconstant coefficients.

Variation of parameters for second order linear DE's

Recall from section 4.2:

- The second order linear differential equation

$$y'' + p(t)y' + q(t)y = g(t)$$

  is equivalent to the system of first order linear differential equations:

$$\mathbf{x}' = \begin{pmatrix} 0 & 1 \\ -q(t) & -p(t) \end{pmatrix} \mathbf{x} + \begin{pmatrix} 0 \\ g(t) \end{pmatrix}$$

  where $\mathbf{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} y(t) \\ y'(t) \end{pmatrix}$.

- The functions y_1 and y_2 form a fundamental set of solutions of

$$y'' + p(t)y' + q(t)y = 0$$

  on the open interval I if and only if $\mathbf{x}_1(t) = \begin{pmatrix} y_1(t) \\ y_1'(t) \end{pmatrix}$ and $\mathbf{x}_2(t) = \begin{pmatrix} y_2(t) \\ y_2'(t) \end{pmatrix}$ form a fundamental set of solutions on I of

$$\mathbf{x}' = \begin{pmatrix} 0 & 1 \\ -q(t) & -p(t) \end{pmatrix} \mathbf{x}\,.$$

- $Y(t)$ is a particular solution of $\quad y'' + p(t)y' + q(t)y = g(t)$
  if and only if $\mathbf{x}_p = \begin{pmatrix} Y(t) \\ Y'(t) \end{pmatrix}$ is a particular solution of

$$\mathbf{x}' = \begin{pmatrix} 0 & 1 \\ -q(t) & -p(t) \end{pmatrix} \mathbf{x} + \begin{pmatrix} 0 \\ g(t) \end{pmatrix}$$

Conclusion:
The first component of $\mathbf{x}_p$ from the method of variations of parameters is a particular solution $Y(t)$ of the 2nd order differential equation.

Explicitly:
Recall that the Wronskian of y_1 and y_2 is

$$W[y_1, y_2](t) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

Theorem (Theorem 4.7.2)

Suppose that p, q and g are continuous on the open interval I and that y_1 and y_2 form a fundamental system of solutions of the homogenous differential equation $y'' + p(t)y' + q(t)y = 0$. Then a particular solution of

$$y'' + p(t)y' + q(t)y = g(t)$$

is:

$$Y(t) = -y_1(t) \int \frac{y_2(t)g(t)}{W[y_1, y_2](t)}\, dt + y_2(t) \int \frac{y_1(t)g(t)}{W[y_1, y_2](t)}\, dt$$

The general solution of $y'' + p(t)y' + q(t)y = g(t)$ is

$$y(t) = c_1 y_1(t) + c_2 y_2(t) + Y(t)$$

where c_1, c_2 are arbitrary constants.

Example: Find a particular solution of the equation $y'' - 3y' - 4y = e^{-t}$.