EXAMPLES (SECTION 4.7)

Find the general solution of the system of DE's : $\mathbf{x}'(t) = \underbrace{\begin{pmatrix} 1 & -1 \\ 2 & 4 \end{pmatrix}}_{A} \mathbf{x}(t) + \underbrace{\begin{pmatrix} t \\ 2t \end{pmatrix}}_{\mathbf{g}(t)}$

The general solution is of the form

$$\mathbf{x}(t) = C_1 \mathbf{x}_1(t) + C_2 \mathbf{x}_2(t) + \mathbf{x}_p(t)$$

where $\mathbf{x}_1$ and $\mathbf{x}_2$ form a fundamental system of solutions of $\mathbf{x}' = A\mathbf{x}$ and $\mathbf{x}_p$ is a particular solution of the given nonhomogeneous system:

- characteristic equation of A: $\begin{vmatrix} 1-\lambda & -1 \\ 2 & 4-\lambda \end{vmatrix} = 0$ i.e. $\lambda^2 - 5\lambda + 6 = 0$

  Two distinct real eigenvalues $\lambda_1 = 2$, $\lambda_2 = 3$.

  Eigenvectors for $\lambda_1 = 2$: $\begin{pmatrix} -1 & -1 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ i.e. $x_1 + x_2 = 0$, i.e. $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = C \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ with $C \neq 0$

  Set $\mathbf{v}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$. Hence $\mathbf{x}_1(t) = e^{2t} \mathbf{v}_1$.

  Eigenvectors for $\lambda_2 = 3$: $\begin{pmatrix} -2 & -1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, i.e. $2x_1 + x_2 = 0$, i.e. $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = C \begin{pmatrix} 1 \\ -2 \end{pmatrix}$ with $C \neq 0$

  Set $\mathbf{v}_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$. Hence $\mathbf{x}_2(t) = e^{3t} \mathbf{v}_2$.

- $X(t) = \begin{pmatrix} e^{2t} & e^{3t} \\ -e^{2t} & -2e^{3t} \end{pmatrix}$ invertible with $W[\mathbf{x}_1, \mathbf{x}_2](t) = e^{5t} \begin{vmatrix} 1 & 1 \\ -1 & -2 \end{vmatrix} = -e^{5t}$

$$X(t)^{-1} = \frac{1}{W[\mathbf{x}_1, \mathbf{x}_2](t)} \begin{pmatrix} -2e^{3t} & -e^{3t} \\ e^{2t} & e^{2t} \end{pmatrix} = e^{-5t} \begin{pmatrix} 2e^{3t} & e^{3t} \\ -e^{2t} & -e^{2t} \end{pmatrix} = \begin{pmatrix} 2e^{-2t} & e^{-2t} \\ -e^{-3t} & -e^{-3t} \end{pmatrix}$$

$$X(t)^{-1} \mathbf{g}(t) = \begin{pmatrix} 2e^{-2t} & e^{-2t} \\ -e^{-3t} & -e^{-3t} \end{pmatrix} \begin{pmatrix} t \\ 2t \end{pmatrix} = \begin{pmatrix} 4te^{-2t} \\ -3te^{-3t} \end{pmatrix}$$

$$\mathbf{x}_p(t) = X(t) \int X^{-1}(t) \mathbf{g}(t)\, dt = \begin{pmatrix} e^{2t} & e^{3t} \\ -e^{2t} & -2e^{3t} \end{pmatrix} \int \begin{pmatrix} 4te^{-2t} \\ -3te^{-3t} \end{pmatrix} dt$$

$$\left[\begin{array}{l} \int te^{-at}\, dt = -\frac{t}{a} e^{-at} + \frac{1}{a} \int e^{-at}\, dt = -\frac{t}{a} e^{-at} - \frac{1}{a^2} e^{-at} + C = \frac{-e^{-at}}{a} \left(t + \frac{1}{a}\right) + C \\ (a \neq 0) \end{array} \right]$$

$$= \begin{pmatrix} e^{2t} & e^{3t} \\ -e^{2t} & -2e^{3t} \end{pmatrix} \begin{pmatrix} -2e^{-2t}\left(t + \frac{1}{2}\right) \\ e^{-3t}\left(t + \frac{1}{3}\right) \end{pmatrix} = \begin{pmatrix} -2\left(t + \frac{1}{2}\right) + \left(t + \frac{1}{3}\right) \\ 2\left(t + \frac{1}{2}\right) - 2\left(t + \frac{1}{3}\right) \end{pmatrix} = \begin{pmatrix} -t - 2/3 \\ 1/3 \end{pmatrix}$$

Conclusion: general solution is

$$\mathbf{x}(t) = C_1 e^{2t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + C_2 e^{3t} \begin{pmatrix} 1 \\ -2 \end{pmatrix} + \begin{pmatrix} -t - 2/3 \\ 1/3 \end{pmatrix}$$

EXAMPLE:

Find a particular solution of $y''-3y'-4y=\underbrace{e^{-t}}_{g(t)}$ using the method of variations of parameters.

The functions $y_1(t)=e^{4t}$ and $y_2(t)=e^{-t}$ form a fundamental system of solutions of $y''-3y'-4y=0$ (on $I=\mathbb{R}$); see section 4.5. Their Wronskian is

$$W[y_1,y_2](t)=\begin{vmatrix} e^{4t} & e^{-t} \\ 4e^{4t} & -e^{-t} \end{vmatrix} = -e^{3t}-4e^{3t} = -5e^{3t}.$$

By the method of variation of parameters, a particular solution of $y''-3y'-4y=e^{-t}$ is

$$Y(t) = -y_1(t)\int \frac{y_2(t)\,g(t)}{W[y_1,y_2](t)}\,dt + y_2(t)\int \frac{y_1(t)\,g(t)}{W[y_1,y_2](t)}\,dt$$

$$= -e^{4t}\int \frac{e^{-2t}}{(-5e^{3t})}\,dt + e^{-t}\int \frac{e^{3t}}{(-5e^{3t})}\,dt =$$

$$= \frac{1}{5}e^{4t}\int e^{-5t}\,dt - \frac{1}{5}te^{-t} = -e^{4t} - \frac{1}{5}te^{-t}$$

Since $y_1(t)=e^{4t}$, also $Y_1(t)=-\frac{1}{5}te^{-t}$ is a particular solution (the one found in section 4.5 with the method of undetermined constants).