Section 5.1: Definition of the Laplace transform

Main Topics:

- **Laplace transform of a function**
- **Piecewise continuity**
- **Examples**
- **Existence theorems**

Pierre-Simon, marquis de Laplace (1745-1827).

Portrait de Paulin Guérin, château de Versailles.

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Improper integrals

Definition

Suppose $a \in \mathbb{R}$ and f is a function defined on the interval $[a, +\infty)$.

The **improper integral** of f from a to $+\infty$, denoted $\int_a^{+\infty} f(t)\,dt$, is defined as the limit

$$\int_a^{+\infty} f(t)\,dt = \lim_{A \to +\infty} \int_a^A f(t)\,dt$$

- If $\int_a^A f(t)\,dt$ exists for each $A > a$ and the above limit exists and is finite, then we say that the improper integral **converges**.
- Otherwise, we say that the improper integral **diverges**.

Example:

- Compute $\int_1^{+\infty} \frac{1}{t}\,dt$

Examples (continued):

- Compute $\displaystyle\int_1^{+\infty} \frac{1}{t}\,dt.$

$$\int_1^{+\infty} \frac{1}{t}\,dt = \lim_{A\to+\infty} \int_1^{A} \frac{1}{t}\,dt = \lim_{A\to+\infty} (\ln(A) - \ln(1)) = +\infty.$$

Conclusion: the improper integral diverges.

- Compute $\displaystyle\int_1^{+\infty} \frac{1}{t^p}\,dt$, where p is a real constant $\neq 1$.

$$\int_1^{+\infty} t^{-p}\,dt = \lim_{A\to+\infty} \int_1^{A} t^{-p}\,dt = \lim_{A\to+\infty} \left[\frac{1}{1-p} t^{1-p}\right]_{t=1}^{t=A} = \lim_{A\to+\infty} \frac{1}{1-p}(A^{1-p} - 1)$$

We have $\lim_{A\to+\infty} A^{1-p} = \begin{cases} +\infty & \text{if } 1-p > 0 \\ 0 & \text{if } 1-p < 0 \end{cases}.$

Conclusion:

$$\int_1^{+\infty} t^{-p}\,dt = \begin{cases} +\infty & \text{if } p < 1 \quad \Rightarrow \text{the improper integral diverges} \\ \frac{-1}{1-p} = \frac{1}{p-1} & \text{if } p > 1 \quad \Rightarrow \text{the improper integral converge to } \frac{1}{p-1} \end{cases}.$$

Examples (continued):

- Compute $\int_0^{+\infty} e^{-st}\,dt$, where s is a real number, $s > 0$.

We have

$$\begin{aligned}\int_0^{+\infty} e^{-st}\,dt &= \lim_{A\to+\infty}\int_0^{A} e^{-st}\,dt = \lim_{A\to+\infty}\Big[-\frac{1}{s}e^{-st}\Big]_{t=0}^{t=A} \\ &= \lim_{A\to+\infty}\Big(-\frac{1}{s}e^{-sA}+\frac{1}{s}\Big) = \frac{1}{s}\end{aligned}$$

Conclusion: the improper integral converges to the value $\frac{1}{s}$.

The Laplace transform

Definition

Let f be a function defined on $[0, +\infty)$. The **Laplace transform of** f is the function F defined by

$$F(s) = \int_0^{+\infty} e^{-st} f(t)\, dt$$

for all values s for which this improper integral converges.

The Laplace transform of f will also be denoted by $\mathcal{L}\{f\}$.

Remarks:

- The Laplace transform F (or $\mathcal{L}\{f\}$) of f is a **function** defined on the set D of all values $s \in \mathbb{R}$ for which the defining improper integral converges.
- **Notational conventions:**
  - ◇ t (representing time) is the variable of the given function f
  - ◇ s is the variable of the Laplace transform F or $\mathcal{L}\{f\}$ of f.
- Departing from the usual functional notation, one often writes "$F(s) = \mathcal{L}\{f(t)\}$".
  This means: the function F (which is function of s) is the Laplace transform of f (which is a function of t).
  The proper (but too long to write) notation would be: "given $f = f(t)$, consider $F(s) = \mathcal{L}\{f\}(s)$."

Examples:

- Let $f(t) = e^{at}$, $t \geq 0$ and $a \in \mathbb{R}$. Then

$$\mathcal{L}\{f\}(s) = \int_0^{\infty} e^{at} e^{-st}\, dt = \lim_{A \to +\infty} \int_0^A e^{(a-s)t}\, dt = \lim_{A \to +\infty} \left[\frac{1}{a-s} e^{(a-s)t} \right]_{t=0}^{t=A}$$

$$= \begin{cases} +\infty & \text{if } s \leq a \\ \frac{1}{s-a} & \text{if } s > a \end{cases}$$

Therefore: the Laplace transform $\mathcal{L}\{f\}$ of $f(t) = e^{at}$, with $a \in \mathbb{R}$, is a function defined on $(a, +\infty)$. Moreover, $\mathcal{L}\{f\}(s) = \dfrac{1}{s-a}$ for $s > a$.

On a table of Laplace transforms, the above is usually shortly written as follows:

$$\mathcal{L}\{e^{at}\} = \frac{1}{s-a}, \qquad s > a.$$

- For $a = 0$, we deduce the Laplace transform of the constant function $f(t) = 1$ for all $t \geq 0$:

$$\mathcal{L}\{1\} = \frac{1}{s}, \qquad s > 0.$$

Examples (continued):

- If we replace $a \in \mathbb{R}$ with $a + ib \in \mathbb{C}$ (where $a, b \in \mathbb{R}$), then

$$\lim_{A \to +\infty} e^{(a+ib-s)A} = \lim_{A \to +\infty} e^{(a-s)A} e^{ibA} \quad \begin{cases} \text{does not exist} & \text{if } s \leq a \\ = 0 & \text{if } s > a \end{cases}$$

This gives, with the same computations as in the case $a \in \mathbb{R}$:

$$\mathcal{L}\{e^{(a+ib)t}\}(s) = \frac{1}{s - a - ib}, \qquad s > a$$

Remark: This is often written on a table omitting the “s” variable on the RHS:

$$\mathcal{L}\{e^{(a+ib)t}\} = \frac{1}{s - a - ib}, \qquad s > a.$$

Theorem (Theorem 5.1.2)

Suppose that:

- f_1 *is a function whose Laplace transform exists on the interval* $(a_1, +\infty)$,
- f_2 *is a function whose Laplace transform exists on the interval* $(a_2, +\infty)$.

Then, for any (real or complex) constants c_1, c_2, *the Laplace transform of* $c_1 f_1 + c_2 f_2$ *exists on the interval* $(\max\{a_1, a_2\}, +\infty)$, *and satisfies*

$$\mathcal{L}\{c_1 f_1 + c_2 f_2\} = c_1 \mathcal{L}\{f_1\} + c_2 \mathcal{L}\{f_2\}$$

In particular, the Laplace transform is a linear operator.

Example: Find the Laplace transform of $f(t) = \sin(bt)$.

Recall that $\sin(bt) = \dfrac{e^{ibt} - e^{-ibt}}{2i}$. For $s > 0$ we have $\mathcal{L}\{e^{\pm ibt}\}(s) = \dfrac{1}{s \mp ib}$.

Hence:

$$\begin{aligned}\mathcal{L}\{\sin(bt)\}(s) = \frac{1}{2i}\mathcal{L}\{e^{ibt}\}(s) - \frac{1}{2i}\mathcal{L}\{e^{-ibt}\}(s) &= \frac{1}{2i}\,\frac{1}{s - ib} - \frac{1}{2i}\,\frac{1}{s + ib} \\ &= \frac{1}{2i}\,\frac{(s + ib) - (s - ib)}{(s - ib)(s + ib)} \\ &= \frac{b}{s^2 + b^2}\end{aligned}$$

Piecewise continuous functions

Definition (Definition 5.1.3)

A function f is said to be **piecewise continuous** on an interval $[\alpha, \beta]$ if this interval can be partitioned by a finite number points $\alpha = t_0 < t_1 < \cdots < t_n = \beta$ so that:

1. f is continuous on each open subinterval (t_{j-1}, t_j), and
2. the limits $\lim_{t \to t_{j-1}^+} f(t)$ and $\lim_{t \to t_j^-} f(t)$ exist and are finite for all $j = 1, \ldots, n$

Example:

$$f(t) = \begin{cases} t + 1 & \text{if } t \in [-1, 0) \\ 1/2 & \text{if } t = 0 \\ t^2 & \text{if } t \in (0, 1) \\ 0 & \text{if } t = 1 \end{cases}$$

Partition of $[-1, 1]$ by $-1 < 0 < 1$

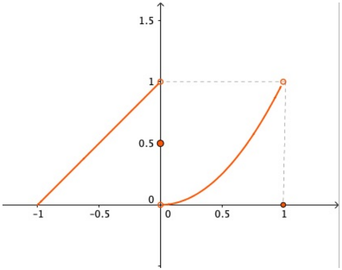
f continuous on $(-1, 0) \cup (0, 1)$

The limits:

$\lim_{t \to -1^+} f(t) = 0\,, \quad \lim_{t \to 0^-} f(t) = 1\,,$

$\lim_{t \to 0^+} f(t) = 0\,, \quad \lim_{t \to 1^-} f(t) = 1$

exist and are finite



f is piecewise continuous on $[-1, 1]$

Example:

$$f(t) = \begin{cases} 1 & \text{if } t \in [-1,0] \\ 1/t & \text{if } t \in (0,1] \end{cases}$$

f is defined for all $t \in [-1,1]$ and continuous on $(-1,0) \cup (0,1)$.
But: $\lim_{t\to 0+} f(t) = \lim_{t\to 0+} 1/t = +\infty$ is not finite.
So f is not piecewise continuous.

Remarks:

- f is piecewise continuous on $[\alpha,\beta]$ provided it is continuous at all but possibly finitely many points of $[\alpha,\beta]$, at each of which f has a jump discontinuity.
- continuous $\Rightarrow$ piecewise continuous, but continuous $\not\Leftarrow$ piecewise continuous
- A similar definition for a piecewise continuous on an open interval (α,β): everything is as above, but you need not check the existence of the limits $\lim_{t\to\alpha^+} f(t)$ and $\lim_{t\to\beta^-} f(t)$.

  This applies in particular if $\alpha = -\infty$ or $\beta = +\infty$.

Example: f piecewise continuous on $(-\infty,\infty)$:

$$f(t) = \begin{cases} -1 & \text{if } t < 0 \\ 0 & \text{if } t = 0 \\ 1 & \text{if } t > 0 \end{cases}$$

Partition of $(-\infty,\infty)$ by $-\infty < 0 < +\infty$
f continuous on $(-\infty,0) \cup (0,+\infty)$
The limits: $\lim_{t\to 0-} f(t) = -1$, $\lim_{t\to 0^+} f(t) = 1$
exist and are finite.

Laplace Transform of a piecewise continuous function

Piecewise continuous $\Rightarrow$ integrable on every finite interval $[\alpha, \beta]$.

Example:

Compute the Laplace transform of $f(t) = \begin{cases} 2t & \text{if } 0 \leq t < 1 \\ 1 & \text{if } t \geq 1 \end{cases}$

Remark: f is piecewise continuous (but not continuous) on $[0, +\infty)$.

$$\mathcal{L}\{f\}(s) = \int_0^{+\infty} f(t)e^{-st}\, dt = 2\int_0^1 te^{-st}\, dt + \int_1^{+\infty} e^{-st}\, dt$$

One computes:

$$\int_0^1 te^{-st}\, dt = \frac{-e^{-s}(s+1)+1}{s} \qquad \text{if } s \neq 0$$

$$\int_1^{+\infty} e^{-st}\, dt = \lim_{A\to+\infty} \int_1^A e^{-st}\, dt = \lim_{A\to+\infty} \left[\frac{-1}{s}e^{-st}\right]_{t=1}^{t=A}$$

$$= \lim_{A\to+\infty} \frac{-1}{s}\left(e^{-sA} - e^{-s}\right) = \frac{e^{-s}}{s} \qquad \text{if } s > 0.$$

Therefore:

$$\mathcal{L}\{f\}(s) = 2\frac{-e^{-s}(s+1)+1}{s} + \frac{e^{-s}}{s} = \frac{-e^{-s}(s+2)+2}{s} \qquad \text{if } s > 0.$$

Tests of convergence of improper integrals

- There are integrals, and hence improper integrals, that cannot be evaluated. We may still be able to determine whether an improper integral converges or not. This is done by comparing it to an improper integral we can compute, e.g. of e^{ct} or t^{-p}.
- If f is a piecewise continuous function on $[a, +\infty)$, then $\int_a^M f(t)\,dt$ is a finite number for all $M \geq a$.

  The convergence/divergence of the improper integral $\int_a^\infty f(t)\,dt$ can be checked from the convergence/divergence of the improper integral $\int_M^\infty f(t)\,dt$.

Theorem (Theorem 5.1.4)

Let a and M be two real numbers so that $M \geq a$. Suppose f is piecewise continuous for $t \geq a$.

- *If $|f(t)| \leq g(t)$ for $t \geq M$ and if $\int_M^{+\infty} g(t)\,dt$ converges, then $\int_a^{+\infty} f(t)\,dt$ converges.*
- *If $f(t) \geq g(t) \geq 0$ for $t \geq M$, and if $\int_M^{+\infty} g(t)\,dt$ diverges then $\int_a^{+\infty} f(t)\,dt$ also diverges.*

Functions of exponential order

Definition (Definition 5.1.5)

A function f is of **exponential order** (as $t \to +\infty$) if there exist real constants $M \geq 0$, $K > 0$ and a such that

$$|f(t)| \leq Ke^{at} \qquad \text{for } t \geq M\,.$$

Remarks:

- The choice of constants M, K, a is not unique.
- To check if f is of exponential order, check if there is a so that $\dfrac{f(t)}{e^{at}}$ is bounded for all large t.

Example:

- $f(t) = e^t$ is of exponential order.
- $f(t) = t^2$ is of exponential order.
- $f(t) = e^{t^2}$ is not of exponential order.

Existence of Laplace transforms

Theorem (Theorem 5.1.6, Corollary 5.1.7)

Suppose:

- f *is piecewise continuous on* $[0, A]$ *for any positive real number* A
- f *is of exponential order, that is* $|f(t)| \leq Ke^{at}$ *for* $t \geq M$.

Then:

(1) *the Laplace transform of* f *exists for* $s > a$,

(2) *there exists a positive constant* L *such that*

$$|\mathcal{L}\{f\}(s)| \leq L/s \qquad \text{for all } s \text{ sufficiently large.}$$

In particular: $\lim_{s \to +\infty} \mathcal{L}\{f\}(s) = 0$.

Example:

Apply the Theorem to $f(t) = t^2$.