Section 5.2: Properties of the Laplace transform

Main Topics:

- **Laplace transform of $e^{ct}f$,**
- **Laplace transform of derivatives**
- **Laplace transform of $t^n f$.**
- **Laplace transform of differential equations.**

Laplace transform of $e^{ct} f$

Recall that the Laplace transform of a function f is defined by

$$\mathcal{L}\{f\}(s) = \int_0^{+\infty} e^{-st} f(t), dt$$

for all values $s \in \mathbb{R}$ for which this integral converges.

Theorem (Theorem 5.2.1)

If the Laplace transform of f exists for $s > a$, then one has

$$\mathcal{L}\{e^{ct} f\}(s) = \mathcal{L}\{f\}(s - c) \qquad \text{for } s > a + c$$

for a constant c.

Example: Find the Laplace transform of $g(t) = e^{-3t} \sin(2t)$.

[Recall: the Laplace transform of $f(t) = \sin(bt)$ is $\mathcal{L}\{f\}(s) = \frac{b}{s^2+b^2}$ for $s > 0$.]

Laplace Transform of derivatives

Recall (Definition 5.1.5) that g is of exponential order if there exist real constants $M \geq 0$, $K > 0$ and a such that $|g(t)| \leq Ke^{at}$ for $t \geq M$.

Theorem (Theorem 5.2.2)

Suppose that:

- *f is continuous on the interval $0 \leq t \leq A$ for all A,*
- *f' is piecewise continuous on the interval $0 \leq t \leq A$ for all A,*
- *f and f' are of exponential order with exponent a.*

Then the Laplace transform of f' exists for $s > a$ and is given by

$$\mathcal{L}\{f'\}(s) = s\mathcal{L}\{f\}(s) - f(0).$$

Example:

- Compute the Laplace transform of $g(t) = \cos(bt)$ where $b \in \mathbb{R}$, $b \neq 0$.

  [Recall that $\mathcal{L}\{e^{\pm ibs}\}(s) = \frac{1}{s \mp b}$ for $s > 0$.]
- Verify the formula $\mathcal{L}\{f'\}(s) = s\mathcal{L}\{f\}(s) - f(0)$ for $f(t) = \sin(bt)$.

Since $f'' = (f')'$ and more generally $f^{(n)} = (f^{(n-1)})'$, we may use induction to generalize the previous theorem to higher derivatives.

Example:

If f', f'' satisfy the same conditions of f, f' in Theorem 5.2.2, then for $s > a$

$$\begin{aligned}
\mathcal{L}\{f''\}(s) &= s\mathcal{L}\{f'\}(s) - f'(0) \\
&= s\big[s\mathcal{L}\{f\}(s) - f(0)\big] - f'(0) \\
&= s^2\mathcal{L}\{f\}(s) - sf(0) - f'(0)\,.
\end{aligned}$$

Corollary (Corollary 5.2.3)

Suppose that

- $f, f', \dots, f^{(n-1)}$ *are continuous on any interval* $0 \le t \le A$,
- $f^{(n)}$ *is piecewise continuous on any interval* $0 \le t \le A$,
- $f, f', \dots, f^{(n)}$ *are of exponential order, with exponent a.*

Then the Laplace transform of $f^{(n)}$ *exists for* $s > a$ *and is given by*

$$\mathcal{L}\{f^{(n)}\}(s) = s^n\mathcal{L}\{f\}(s) - s^{(n-1)}f(0) - \cdots - sf^{(n-2)}(0) - f^{(n-1)}(0)$$

Laplace Transform of $t^n f(t)$

Theorem (Theorem 5.2.4)

Suppose

- f *is piecewise continuous on any interval* $0 \leq t \leq A$,
- f *is of exponential order with* $|f^{(n)}(t)| \leq Ke^{at}$.

Then for any positive integer n, we have

$$\mathcal{L}\{t^n f\}(s) = (-1)^n (\mathcal{L}\{f\})^{(n)}(s) \qquad \text{for } s > a.$$

Example:

Compute $\mathcal{L}\{t\}$, $\mathcal{L}\{t^2\}$ and $\mathcal{L}\{2t^2 - 3t + 1\}$.

[Recall that $\mathcal{L}\{1\}(s) = \dfrac{1}{s}$ for $s > 0$.]

Laplace Transform of $t^n f(t)$

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Example:

Compute $\mathcal{L}\{t\}$, $\mathcal{L}\{t^2\}$ and $\mathcal{L}\{2t^2 - 3t + 1\}$.

[Recall that $\mathcal{L}\{1\}(s) = \dfrac{1}{s}$ for $s > 0$.]

We have $\left(\dfrac{1}{s}\right)' = -\dfrac{1}{s^2}$ and $\left(\dfrac{1}{s}\right)'' = \dfrac{2}{s^3}$. Hence for $s > 0$:

$$\mathcal{L}\{t\}(s) = (-1)(\mathcal{L}\{1\})'(s) = \frac{1}{s^2} \quad \text{and} \quad \mathcal{L}\{t^2\}(s) = (-1)^2 \mathcal{L}\{1\})''(s) = \frac{2}{s^3}$$

Therefore:

$$\mathcal{L}\{2t^2 - 3t + 1\} = 2\mathcal{L}\{t^2\} - 3\mathcal{L}\{t\} + \mathcal{L}\{1\} = \frac{4}{s^3} + 3\frac{1}{s^2} + \frac{1}{s} \qquad \text{for } s > 0.$$

By

$$\left(\frac{1}{s}\right)^{(n)} = (-1)^n \frac{n!}{s^{n+1}} \,,$$

we get the following corollary:

Corollary (Corollary 5.2.5)

For any positive integer n, we have

$$\mathcal{L}\{t^n\}(s) = \frac{n!}{s^{n+1}} \qquad \text{for } s > 0 \,.$$

Laplace Transform of differential equations

Compute the Laplace transform $Y(s)$ of the solution $y(t)$ of the differential equation

$$y'' + 3y' + 2y = e^{-3t}$$

which satisfies with initial conditions $y(0) = 1$, $y'(0) = 0$.

[Assume that y satisfies the hypothesis of Corollary 5.2.3 to compute $Y = \mathcal{L}\{y\}$, $\mathcal{L}\{y'\}$ and $\mathcal{L}\{y''\}$.]