Section 5.3: Inverse Laplace transform

From Section 5.1: applying the Laplace transform to the IVP

$$y'' + ay' + by = f(t) \qquad \text{with initial conditions } y(0) = y_0,\ y'(0) = y_1$$

leads to an algebraic equation for $Y = \mathcal{L}\{y\}$, where $y(t)$ is the solution of the IVP.

The algebraic equation can be solved for $Y = \mathcal{L}\{y\}$.

We now want to determine y out $Y = \mathcal{L}\{y\}$.

This is equivalent to inverting the Laplace transform and find $y = \mathcal{L}^{-1}\{Y\}$.

Main Topics:

- **Inverse Laplace transform**
- **Integrals of partial fractions**
- **Tables of Laplace transforms**

Inverse Laplace transform of a piecewise continuous function

Theorem (Existence of the inverse Laplace transform)

Suppose that $f(t)$ and $g(t)$ are piecewise continuous and of exponential order on $[0, +\infty)$.
If $\mathcal{L}\{f\} = \mathcal{L}\{g\}$, then $f = g$ at all points where f and g are continuous.

This theorem allows us to define in an essentially unique way the inverse Laplace transform.

Definition (Definition 5.3.2)

Let f be a piecewise continuous and of exponential order on $[0, +\infty)$.
If $F = \mathcal{L}\{f\}$, then f is called **inverse Laplace transform** of F, and we write it $f = \mathcal{L}^{-1}(F)$.

The computation of the inverse transform of a function requires advanced tools. In this class we will focus on the cases which are covered by Table 5.3.1 together some special properties of the Laplace transform (linearity and the properties in Section 5.2).

From the linearity of the Laplace transform, we obtain the linearity of its inverse:

Theorem (Theorem 5.3.3)

Suppose that $f_1 = \mathcal{L}^{-1}(F_1)$ *and* $f_2 = \mathcal{L}^{-1}(F_2)$ *are piecewise continuous and of exponential order on* $[0, \infty)$. *Then*

$$\mathcal{L}^{-1}(c_1 F_1 + c_2 F_2) = c_1 \mathcal{L}^{-1}(F_1) + c_2 \mathcal{L}^{-1}(F_2)$$

for arbitrary constants c_1 *and* c_2.
In other words, the inverse Laplace transform is a linear operator.

TABLE 5.3.1 Elementary Laplace transforms.

	$f(t) = \mathcal{L}^{-1}\{F(s)\}$	$F(s) = \mathcal{L}\{f(t)\}$	Notes
1.	1	$\frac{1}{s}, \quad s > 0$	Sec. 5.1; Ex. 4
2.	e^{at}	$\frac{1}{s-a}, \quad s > a$	Sec. 5.1; Ex. 5
3.	$t^n, \quad n =$ positive integer	$\frac{n!}{s^{n+1}}, \quad s > 0$	Sec. 5.2; Cor. 5.2.5
4.	$t^p, \quad p > -1$	$\frac{\Gamma(p+1)}{s^{p+1}}, \quad s > 0$	Sec. 5.1; Prob. 37
5.	$\sin at$	$\frac{a}{s^2+a^2}, \quad s > 0$	Sec. 5.1; Ex. 7
6.	$\cos at$	$\frac{s}{s^2+a^2}, \quad s > 0$	Sec. 5.1; Prob. 22
7.	$\sinh at$	$\frac{a}{s^2-a^2}, \quad s > \|a\|$	Sec. 5.1; Prob. 19
8.	$\cosh at$	$\frac{s}{s^2-a^2}, \quad s > \|a\|$	Sec. 5.1; Prob. 18
9.	$e^{at}\sin bt$	$\frac{b}{(s-a)^2+b^2}, \quad s > a$	Sec. 5.1; Prob. 23
10.	$e^{at}\cos bt$	$\frac{s-a}{(s-a)^2+b^2}, \quad s > a$	Sec. 5.1; Prob. 24
11.	$t^n e^{at}, \quad n =$ positive integer	$\frac{n!}{(s-a)^{n+1}}, \quad s > a$	Sec. 5.2; Prob. 8
12.	$u_c(t)$	$\frac{e^{-cs}}{s}, \quad s > 0$	Sec. 5.5; Eq. (4)
13.	$u_c(t)f(t-c)$	$e^{-cs}F(s)$	Sec. 5.5; Eq. (6)
14.	$e^{ct}f(t)$	$F(s-c)$	Sec. 5.2; Thm. 5.2.1
15.	$\int_0^t f(t-\tau)g(\tau)\,d\tau$	$F(s)G(s)$	Sec. 5.6; Thm. 5.8.3
16.	$\delta(t-c)$	e^{-cs}	Sec. 5.7; Eq. (14)
17.	$f^{(n)}(t)$	$s^nF(s) - s^{n-1}f(0) - \cdots - f^{(n-1)}(0)$	Sec. 5.2; Cor. 5.2.3
18.	$t^nf(t)$	$(-1)^nF^{(n)}(s)$	Sec. 5.2; Thm. 5.2.4

Example:

Find the inverse Laplace transform of the following functions:

- $F(s) = \dfrac{2s}{s^2 - 1}$
- $F(s) = \dfrac{3}{s^2 + 2s + 5}$

Partial fraction decomposition

Factor in denominator	Term in partial fraction decomposition
$ax+b$	$\dfrac{A}{ax+b}$
$(ax+b)^k$	$\dfrac{A_1}{ax+b}+\dfrac{A_2}{(ax+b)^2}+\cdots+\dfrac{A_k}{(ax+b)^k}$
ax^2+bx+c (without real roots)	$\dfrac{Ax+B}{ax^2+bx+c}$
$(ax^2+bx+c)^k$ (without real roots)	$\dfrac{A_1x+B_1}{ax^2+bx+c}+\dfrac{A_2x+B_2}{(ax^2+bx+c)^2}+\cdots\dfrac{A_kx+B_k}{(ax^2+bx+c)^k}$

Example:

$$\frac{x}{(x-1)(x-2)} = \frac{A}{x-1}+\frac{B}{x-2}$$

$$\frac{x^2+x+1}{(x-1)^2(x-2)} = \frac{A_1}{x-1}+\frac{A_2}{(x-1)^2}+\frac{B}{x-2}$$

$$\frac{x^2+x+1}{(x^2+1)(x-2)} = \frac{A_1x+B_1}{x^2+1}+\frac{A_2}{x-2}$$

$$\frac{x^2+x+1}{(x^2+1)^2(x-2)} = \frac{A_1x+B_1}{x^2+1}+\frac{A_2x+B_2}{(x^2+1)^2}+\frac{A_3}{x-2}$$