Section 5.4: Solving differential equation with Laplace transforms

Main Topics:

The **Laplace transform method** to solve **initial value problems** for

- **second order linear DEs with constant coefficients**
- **higher order linear DEs with constant coefficients**
- **systems of first order linear DEs with constant coefficients**

The Laplace transform method

From Sections 5.1 and 5.2: applying the Laplace transform to the IVP

$$y'' + ay' + by = f(t) \qquad \text{with initial conditions } y(0) = y_0,\ y'(0) = y_1$$

leads to an algebraic equation for $Y = \mathcal{L}\{y\}$, where $y(t)$ is the solution of the IVP.

The algebraic equation can be solved for $Y = \mathcal{L}\{y\}$.

Inverting the Laplace transform leads to the solution $y = \mathcal{L}^{-1}\{Y\}$.

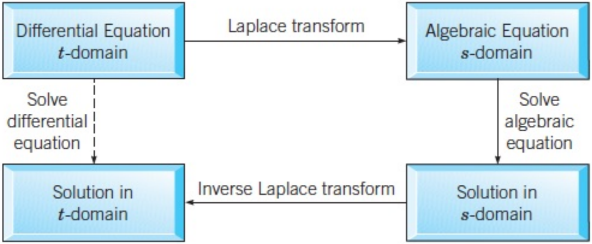

FIGURE 5.0.1 Laplace transform method for solving differential equations.
From: J. Brannan & W. Joyce, Differential equations.

Example:

Solve the IVP: $y'' + 3y' + 2y = e^{-3t}$ with initial conditions $y(0) = 1$, $y'(0) = 0$.

- Apply the Laplace transform to both sides of the DE:

$$\mathcal{L}\{y''\} + 3\mathcal{L}\{y'\} + 2\mathcal{L}\{y\} = \mathcal{L}\{e^{-3t}\}$$

i.e. $$\left[s^2\mathcal{L}\{y\}(s) - sy(0) - y'(0)\right] - 3\left[s\mathcal{L}\{y\}(s) - y(0)\right] + 2\mathcal{L}\{y\}(s) = \frac{1}{s+3}$$

i.e. $$\left[s^2 Y(s) - sy(0) - y'(0)\right] - 3\left[sY(s) - y(0)\right] + 2Y(s) = \frac{1}{s+3}$$

where $Y(s) = \mathcal{L}\{y\}(s)$.

- Insert the initial condition $y(0) = 1$, $y'(0) = 0$:

$$\left[s^2 Y(s) - s\right] - 3\left[sY(s) - 1\right] + 2Y(s) = \frac{1}{s+3}$$

- Solve for $Y(s)$:

$$Y(s) = \frac{s-3}{s^2 - 3s + 2} + \frac{1}{(s+3)(s^2 - 3s + 2)} = \frac{s^2 - 8}{(s+3)(s-2)(s-1)}$$

Remark: $s^2 - 3s + 2$ is the characteristic polynomial of the DE $y'' + 3y' + 2y = 0$.

- Compute $\mathcal{L}^{-1}\{Y\}$ for $Y(s) = \frac{s^2-8}{(s+3)(s-2)(s-1)}$:

(1) *Partial fraction decomposition:*

$$\frac{s^2-8}{(s+3)(s-2)(s-1)} = \frac{A}{s+3} + \frac{B}{s-2} + \frac{C}{s-1}$$

is equivalent to

$$(A+B+C)s^2 = (-3A+2B+C)s + (2A-3B-6C) = s^2 - 8\,.$$

Equating the coefficients of s^2, s and 1 leads to a linear system of equations in A, B, C.

Solution: $A = \frac{1}{20}$, $B = -\frac{4}{5}$, $C = \frac{1}{20}$.

(2) *Linearity of* $\mathcal{L}^{-1}$*:*

$$\mathcal{L}^{-1}\{Y(s)\} = A\mathcal{L}^{-1}\left\{\frac{1}{s+3}\right\} + B\mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\} + C\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\}.$$

(3) Look at the tables to find the inverse Laplace transforms: if $F(s) = \frac{1}{s-a}$ $(s > a)$, then $\mathcal{L}^{-1}\{F\}(t) = e^{at}$.

- **Conclusion:** $y(t) = \mathcal{L}^{-1}\{Y\} = \frac{1}{20}e^{-3t} - \frac{4}{5}e^{2t} + \frac{7}{4}e^{t}$.

In general, taking the Laplace transform of the initial value problem:

$$ay'' + by' + cy = f \quad \text{with} \quad y(0) = y_0 \qquad \text{and} \qquad y'(0) = y_1$$

gives

$$a\big[s^2 Y(s) - sy(0) - y'(0)\big] + b\big[sY(s) - y(0)\big] + cY(s) = F(s)$$

where:

- $Y(s) = \mathcal{L}\{y\}(s)$ is the Laplace transform of y,
- $F(s) = \mathcal{L}\{f\}(s)$ is the Laplace transform of f.

It can be rewritten as

$$(as^2 + bs + c)Y(s) - (as + b)y(0) - ay'(0) = F(s)\,.$$

So,

$$Y(s) = \frac{(as + b)y(0) + ay'(0)}{as^2 + bs + c} + \frac{F(s)}{as^2 + bs + c}\,.$$

The denominator $as^2 + bs + c$ is the characteristic polynomial of $ay'' + by' + cy = f$. (Recall that $as^2 + bs + c = 0$ is its characteristic equation.)

This can be generalized to constant coefficient linear DE of arbitrary order:

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = f(t)$$

with

$$y^{(n-1)}(0) = y_{n-1},\ y^{(n-2)}(0) = y_{n-2},\ \cdots,\ y(0) = y_0\,.$$

The Laplace transform $Y(s) = \mathcal{L}\{y\}(s)$ of the solution $y(t)$ is

$$Y(s) = \frac{(a_n s^{n-1} + \cdots + a_1)y(0) + \cdots + (a_n s + a_{n-1})y^{(n-2)}(0) + a_n y^{(n-1)}(0)}{a_n s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0} + \frac{F(s)}{a_n s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0}$$

where $F(s) = \mathcal{L}\{f\}(s)$ is the Laplace transform of f.

Example: Use the Laplace transform method to solve the IVP

$$y''' + 3y' = \sin(2t) \quad \text{with} \quad y(0) = 1,\ y'(0) = 0,\ y''(0) = -2$$

Laplace transform method for systems of 1st order linear DEs

Consider the system of first order constant coefficient linear DE with initial conditions:

$$\begin{aligned} y_1' &= a_{11}y_1 + a_{12}y_2 + f_1(t) \\ y_2' &= a_{21}y_1 + a_{22}y_2 + f_2(t) \end{aligned}$$

with initial conditions $y_1(0) = y_{10}$, $y_2(0) = y_{20}$.

Take the Laplace transform of each equation and set

$$Y_1 = \mathcal{L}\{y_1\}\,, \quad Y_2 = \mathcal{L}\{y_2\}\,, \quad F_1 = \mathcal{L}\{f_1\}\,, \quad F_2 = \mathcal{L}\{f_2\}\,.$$

We get:

$$\begin{aligned} sY_1 - y_1(0) &= a_{11}Y_1 + a_{12}Y_2 + F_1(t) \\ sY_2 - y_2(0) &= a_{21}Y_1 + a_{22}Y_2 + F_2(t)\,. \end{aligned}$$

This can be rewritten as

$$\begin{aligned} (s - a_{11})Y_1 - a_{12}Y_2 &= y_{10} + F_1(s) \\ -a_{21}Y_1 + (s - a_{22})Y_2 &= y_{20} + F_2(s) \end{aligned}$$

Matrix reformulation:

$$(s\mathbf{I} - \mathbf{A})\mathbf{Y} = \mathbf{y}_0 + \mathbf{F}(s)$$

with

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \qquad \mathbf{F}(s) = \begin{pmatrix} F_1(s) \\ F_2(s) \end{pmatrix}, \qquad \mathbf{y}_0 = \begin{pmatrix} y_{10} \\ y_{20} \end{pmatrix}, \qquad \mathbf{Y} = \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix}$$

If the matrix $s\mathbf{I} - \mathbf{A}$ is invertible, then we can solve for $\mathbf{Y}$:

$$\mathbf{Y} = (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{y}_0 + (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{F}(s)$$

The solution of of the given IVP is:

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} \mathcal{L}\{Y_1\} \\ \mathcal{L}\{Y_2\} \end{pmatrix}.$$

Example: Use the Laplace transform method to solve the initial value problem

$$\begin{aligned} y_1' &= -y_1 + y_2 + e^t \\ y_2' &= y_1 - y_2 + e^t \end{aligned}$$

with $y_1(0) = 1$ and $y_2(0) = 1$