Section 5.5: Discontinuous functions and periodic functions

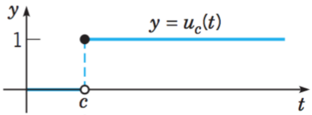
Main Topics:

- **Unit step functions,**
- **indicator functions,**
- **translates of functions,**
- **periodic functions,**
- **and their Laplace transforms.**

Unit step functions

Definition

For a real number c, the **unit step function** u_c is defined by $u_c(t) = \begin{cases} 0 & \text{if } t < c \\ 1 & \text{if } t \geq c \end{cases}$.



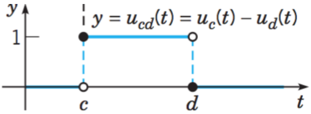
- $u_c(t)$ is piecewise continuous but not continuous.
- When $c = 0$, the unit step function is known as the **Heaviside function**.
- The definition of the value of u_c at the jump discontinuity $t = c$ is immaterial. We could just not define it at all at c.

Indicator functions

Definition

For real numbers $c < d$, the **indicator function for the interval** $[c, d)$ is the function u_{cd} defined by

$$u_{cd}(t) = \begin{cases} 0 & \text{if } t < c \text{ or } t \geq d \\ 1 & \text{if } c \leq t < d \end{cases}$$



- u_{cd} is piecewise continuous but is not continuous.
- As for u_c, the value of u_{cd} at the jump discontinuities is immaterial. We could just not define them at all at c and d.

Representation of piecewise continuous functions

Example:
(1) Use the unit step functions to give a representation of the piecewise continuous function

$$f(t) = \begin{cases} t & \text{if } 0 \leq t < 2 \\ 1 & \text{if } 2 \leq t < 3 \\ e^{-2t} & \text{if } t \geq 3 \end{cases}$$

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$$\begin{aligned} f(t) &= t \cdot u_{02}(t) + 1 \cdot u_{23}(t) + e^{-2t} \cdot u_3(t) \\ &= t\,(u_0(t) - u_2(t)) + (u_2(t) - u_3(t)) + e^{-2t} u_3(t) \\ &= t u_0(t) - (t-1) u_2(t) + (e^{-2t} - 1) u_3(t) \end{aligned}$$

Since $u_0(t) = 1$ for all $t \geq 0$, when we restrict ourselves to $t \geq 0$, we can write:

$$f(t) = t - (t-1)u_2(t) + (e^{-2t} - 1)u_3(t) \qquad (t \geq 0)\,.$$

Laplace transforms of u_c and u_{cd}

- For $s > 0$

$$\mathcal{L}\{u_c\}(s) = \frac{e^{-cs}}{s}$$

Indeed,

$$\begin{aligned}
\mathcal{L}\{u_c\}(s) &= \int_0^\infty e^{-st} u_c(t)\, dt = \int_c^\infty e^{-st}\, dt = \lim_{A \to +\infty} \int_c^A e^{-st}\, dt \\
&= \lim_{A \to +\infty} \left(\frac{e^{-cs}}{s} - \frac{e^{-cA}}{s} \right) = \frac{e^{-cs}}{s}.
\end{aligned}$$

Remark: For $s > 0$ we have $\mathcal{L}\{u_0\}(s) = \dfrac{1}{s} = \mathcal{L}\{1\}(s)$.
This is correct because $u_0(t) = 1$ for $t > 0$.

- For $s > 0$

$$\mathcal{L}\{u_{cd}\}(s) = \mathcal{L}\{u_c\}(s) - \mathcal{L}\{u_d\}(s) = \frac{e^{-cs} - e^{-ds}}{s}.$$

Translate of a function

Definition

Fix $c \geq 0$ a real number and let f be a function defined for $t \geq 0$. The **translate of** f is the function g defined by

$$g(t) = \begin{cases} 0 & \text{if } t < c \\ f(t-c) & \text{if } t \geq c \end{cases} = u_c(t)f(t-c)$$

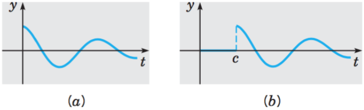


FIGURE 5.5.3 A translation of the given function. (*a*) $y = f(t)$; (*b*) $y = u_c(t)f(t-c)$.

Theorem (Theorem 5.5.1)

Suppose $\mathcal{L}\{f\}(s)$ exists for $s > a \geq 0$. Let $c \geq 0$. Then for $s > a \geq 0$,

$$\mathcal{L}\{u_c(t)f(t-c)\}(s) = e^{-cs}\mathcal{L}\{f\}(s)\,.$$

Example:

Compute the Laplace transform of $f(t) = \begin{cases} t & \text{if } 0 < t < 2 \\ 1 & \text{if } 2 \le t < 3 \\ e^{-2t} & \text{if } t \ge 3 \end{cases}$.

Example:

Compute the Laplace transform of $f(t) = \begin{cases} t & \text{if } 0 < t < 2 \\ 1 & \text{if } 2 \leq t < 3 \\ e^{-2t} & \text{if } t \geq 3 \end{cases}$.

We have found

$$f(t) = tu_0(t) - (t-1)u_2(t) + (e^{-2t} - 1)u_3(t)\,.$$

This can be rewritten as:

$$f(t) = u_0(t)f_1(t-0) - u_2(t)f_2(t-2) + u_3(t)f_3(t-3)$$

where

$$f_1(t) = t, \qquad f_2(t) = t+1, \qquad f_3(t) = e^{-2(t+3)} - 1 = e^{-6}e^{-2t} - 1.$$

Hence for $s > 0$:

$$\begin{aligned} \mathcal{L}\{f\}(s) &= \mathcal{L}\{u_0(t)f_1(t-0)\}(s) - \mathcal{L}\{u_2(t)f_2(t-2)\}(s) + \mathcal{L}\{u_3(t)f_3(t-3)\}(s) \\ &= \mathcal{L}\{f_1(t)\}(s) + e^{-2s}\mathcal{L}\{f_2(t)\}(s) + e^{-3s}\mathcal{L}\{f_3(t)\}(s) \\ &= \frac{1}{s^2} - e^{-2s}\frac{1+s}{s^2} + e^{-3s}\left(\frac{e^{-6}}{s+2} - \frac{1}{s}\right) \end{aligned}$$

(2) Express the inverse Laplace transform f of

$$F(s) = \frac{1 - e^{-2s}}{s^2}$$

in terms of a unit step function.
We have

$$\begin{aligned} \mathcal{L}^{-1}\{F(s)\} &= \mathcal{L}^{-1}\{\frac{1}{s^2}\} - \mathcal{L}^{-1}\{\frac{e^{-2s}}{s^2}\} \\ &= t - u_2(t)(t-2) \end{aligned}$$

Hence

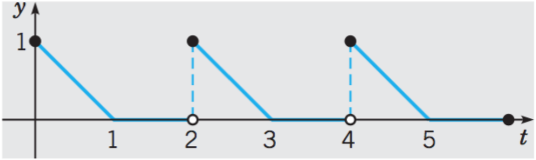
$$f(t) = \begin{cases} t & \text{if } 0 \le t < 2 \\ 2 & \text{if } t \ge 2 \end{cases}$$

Periodic functions

Definition (Definition 5.5.2)

A function f is said to be periodic with period $T > 0$ if $f(t+T) = f(t)$ for all t in the domain of f.

- The sine and cosine functions are 2π-periodic, while the tangent function is π-periodic.
- $f(t) = \begin{cases} 1-t & \text{if } 0 \leq t < 1 \\ 0 & \text{if } 1 \leq t < 2 \end{cases}$ can be turned into a 2-periodic function as follows:



A key property of T-periodic functions is that they can be studied only on any interval of length T. For this, it is convenient to introduce the **window function** $f_T(t)$ associated with f:

$$f_T(t) = f(t)(1 - u_T(t)) = \begin{cases} f(t) & \text{if } 0 \leq t \leq T \\ 0 & \text{otherwise} \end{cases}$$

Write $F_T(s) = \mathcal{L}\{f_T\}(s)$ for the Laplace transform of f_T:

$$F_T(s) = \int_0^\infty f_T(t)e^{-st}\,dt = \int_0^T f(t)e^{-st}\,dt$$

Theorem (Theorem 5.5.3)

If f is T-periodic and piecewise continuous on $[0, T]$, then $\mathcal{L}\{f\}(s) = \dfrac{F_T(s)}{1 - e^{-sT}}$

Example:

Compute the Laplace transform of the 2-periodic function f defined on $[0, 2)$ by

$$f(t) = \begin{cases} t & \text{if } 0 \leq t < 1 \\ 0 & \text{if } 1 \leq t < 2 \end{cases}$$

Here $T = 2$ and $F_2(s) = \displaystyle\int_0^2 f_T(t)dt = \int_0^1 te^{-st}dt = \frac{1 - e^{-s}}{s^2} - \frac{e^{-s}}{s}$.

So, $F(s) = \mathcal{L}\{f\}(s) = \dfrac{1 - e^{-s}}{s^2(1 - e^{-2s})} - \dfrac{e^{-s}}{s(1 - e^{-2s})}$.

Example:

Compute the inverse Laplace transform of the function $F(s) = \dfrac{1}{s(1 + e^{-s})}$