# EXAMPLE (SECTION 5.6)

$y'' + \pi^2 y = f(t)$, $y(0)=0$, $y'(0)=0$ where $f$ is the 2-periodic function defined by $f(t) = \begin{cases} 1 & \text{if } 0 \le t < 1 \\ 0 & \text{if } 1 \le t < 2 \end{cases}$

Apply $\mathcal{L}$ to both sides and set $Y = \mathcal{L}\{y\}$, $F = \mathcal{L}\{f\}$

$$\mathcal{L}\{y''\}(s) + \pi^2 \mathcal{L}\{y\}(s) = F(s)$$

$$s^2 Y(s) - s \underbrace{y(0)}_{=0} - \underbrace{y'(0)}_{=0} + \pi^2 Y(s) = F(s)$$

$$Y(s) = \frac{1}{s^2+\pi^2} F(s)$$

We computed: $F(s) = \dfrac{1}{s(1+e^{-s})}$ $(s>0)$ where $\dfrac{1}{1+e^{-s}} = \sum_{k=0}^{\infty} (-1)^k e^{-ks}$ (geometric series)

$$= \frac{1}{s(s^2+\pi^2)} \sum_{k=0}^{\infty} (-1)^k e^{-ks} \quad (*)$$

Partial fraction decomposition: $\dfrac{1}{s(s^2+\pi^2)} = \dfrac{A}{s} + \dfrac{Bs+C}{s^2+\pi^2} = \dfrac{(A+B)s^2 + Cs + A\pi^2}{s(s^2+\pi^2)}$

$$\Leftrightarrow (A+B)s^2 + Cs + A\pi^2 = 1 \Leftrightarrow \begin{cases} A+B=0 \\ C=0 \\ A\pi^2 = 1 \end{cases} \Rightarrow \begin{array}{l} B = -A = -1/\pi^2 \\ C = 0 \\ A = 1/\pi^2 \end{array}$$

Hence: $\dfrac{1}{s(s^2+\pi^2)} = \dfrac{1}{\pi^2}\left[\dfrac{1}{s} - \dfrac{s}{s^2+\pi^2}\right]$ and

$$\frac{1}{s} - \frac{s}{s^2+\pi^2} = \underset{(s>0)}{\mathcal{L}\{1\}(s)} - \underset{(s>0)}{\mathcal{L}\{\cos(\pi t)\}(s)} = \underset{(s>0)}{\mathcal{L}\{1-\cos(\pi t)\}(s)}$$

Thus $(*)$ becomes

$$Y(s) = \frac{1}{\pi^2} \sum_{k=0}^{\infty} (-1)^k \left[\frac{1}{s} - \frac{s}{s^2+\pi^2}\right] e^{-ks}$$

$$\underset{(s>0)}{=} \frac{1}{\pi^2} \sum_{k=0}^{\infty} (-1)^k \underbrace{\mathcal{L}\{1-\cos(\pi t)\}(s) e^{-ks}}_{\mathcal{L}\{u_k(t)(1-\cos(\pi(t-k)))\}(s)}$$

$[c>0]$
$\mathcal{L}\{u_c(t) f(t-c)\}(s) = e^{-cs}\mathcal{L}\{f\}(s)$
for all $s$ where $\mathcal{L}\{f\}$ is def

$$\cos(\pi(t-k)) = \cos(\pi t - \pi k) = \cos(\pi t)\underbrace{\cos(\pi k)}_{(-1)^k} + \sin(\pi t)\underbrace{\sin(\pi k)}_{=0} = (-1)^k \cos(\pi t)$$

$$Y(s) = \frac{1}{\pi^2}\sum_{k=0}^{\infty} (-1)^k \mathcal{L}\{u_k(t)(1-(-1)^k\cos(\pi t))\}(s) \qquad \text{for } s>0$$

$$= \frac{1}{\pi^2}\sum_{k=0}^{\infty} \mathcal{L}\{u_k(t)\,(\,(-1)^k - \underbrace{(-1)^k(-1)^k}_{=1}\cos(\pi t))\}(s)$$

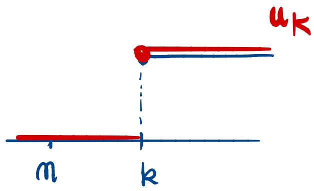
$$= \frac{1}{\pi^2}\sum_{k=0}^{\infty} \mathcal{L}\{((-1)^k - \cos(\pi t))\,u_k(t)\}(s)$$

REMARK: $g_k(t)$ function def on $[0,+\infty)$ $\forall k$

Then at each $t\in[0,+\infty)$ $\quad \sum_{k=0}^{\infty} g_k(t)u_k(t)$ is a finite sum

Indeed: if $t\in[0,m]$, then $u_k(t)=0$ for $k>m$

Hence $\quad \sum_{k=0}^{\infty} g_k(t)\,u_k(t) = \sum_{k=0}^{m} g_k(t)\,u_k(t)$ if $t\in[0,m]$




$$= \mathcal{L}\left\{\frac{1}{\pi^2}\sum_{k=0}^{\infty}((-1)^k-\cos(\pi t))\,u_k(t)\right\}(s)$$

Thus

$$y(t) = \mathcal{L}^{-1}\{Y(s)\}(t) = \frac{1}{\pi^2}\sum_{k=0}^{\infty}((-1)^k-\cos(\pi t))\,u_k(t) \qquad \forall t\geq 0$$

This is a continuous function because it is continuous on every interval $[0,N)$. To see this: for $t\in[0,N)$

$$y(t) = \frac{1}{\pi^2}\sum_{k=0}^{N}((-1)^k-\cos(\pi k))\,u_k(t)$$

→ continuous on $(0,1)\cup(1,2)\cup\ldots\cup(0,N)$ since all $u_k$ have disc. only at integers

At $t\in(m-1,m)$

$$y(t) = \frac{1}{\pi^2}\sum_{k=0}^{m}((-1)^k-\cos(\pi t))\,u_k(t) = \frac{1}{\pi^2}\Big[\underbrace{\sum_{k=0}^{m-1}((-1)^k-\cos(\pi t))\,u_k(t)}_{\sum_{k=0}^{m-1}((-1)^k-\cos(\pi t))}+((-1)^m-\cos(\pi t))\,u_m(t)\Big]$$

$$\lim_{y\to m^-} y(t) = \frac{1}{\pi^2}\Big[\sum_{k=0}^{m-1}((-1)^k-\cos(\pi m))\Big] \qquad \text{because } \lim_{t\to m^-} u_m(t)=0$$

$$\lim_{y\to m^+} y(t) = \frac{1}{\pi^2}\Big[\sum_{k=0}^{m-1}((-1)^k-\cos(\pi m))\Big] + \underbrace{\frac{1}{\pi^2}\Big[(-1)^m - \underbrace{\cos(\pi m)}_{(-1)^m}\Big]}_{=0}$$

Thus $\lim_{y\to m^-} = \lim_{y\to m^+}$

i.e. continuous.

Value at $t = n$

$$y(n) = \lim_{y \to n^+} y(t) = \frac{1}{\pi^2}\left[\sum_{k=0}^{n-1} \left((-1)^k - \cos(\pi n)\right)\right]$$

$$= \frac{1}{\pi^2}\left(\underbrace{\sum_{k=0}^{n-1} (-1)^k} - n \overbrace{\cos(\pi n)}^{(-1)^n}\right)$$

$$\left[\begin{array}{lll} (1-1) + (1-1) + \dots + (-1)^{n-1} & = 1 & \text{if } n-1 \text{ even} \\ (1-1) + (1-1) + \dots + (1-(-1)^{n-1}) & = 0 & \text{if } n \text{ odd} \end{array}\right]$$

$n-1$ even $\Rightarrow$ $n$ odd $\Rightarrow (-1)^n = -1$
$n-1$ odd $\Rightarrow$ $n$ even $\Rightarrow (-1)^n = 1$

$$= \begin{cases} \frac{1}{\pi^2}\left[1 - n(-1)\right] = \frac{n+1}{\pi^2} & \text{if } n-1 \text{ even (ie } n \text{ odd)} \\ \frac{1}{\pi^2}\left[0 - n \cdot 1\right] = -\frac{n}{\pi^2} & \text{if } n-1 \text{ odd (i.e. } n \text{ even)} \end{cases}$$

$\rightsquigarrow$ oscillations at larger and larger amplitudes.