Section 5.7: Impulse functions

Main Topics:

- **Impulse functions,**
- **Dirac's delta,**
- **and thier Laplace transform**

In some applications, one needs to model forced systems where the external force has large magnitude but acts for a very short time. **e.g.:** a mass hit by a hammer

If $g(t)$ is the function describing the external force at time t and the action starts at $t = t_0$ and lasts for a small time $\epsilon > 0$, then:

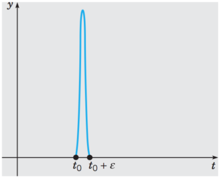
- $g(t)$ is large when t belongs to $[t_0, t_0 + \epsilon)$;
- $g(t) = 0$ outside this interval.

Definition

The total **impulse** of the force $g(t)$ over the interval $[t_0, t_0 + \epsilon)$ is the integral

$$I(\epsilon) = \int_{t_0}^{t_0+\epsilon} g(t)dt = \int_{-\infty}^{+\infty} g(t)dt \,.$$

The total impulse is the area of the region between the bell and the t-axis.



A force of large magnitude active during the short time interval $[t_0, t_0 + \epsilon)$.

Definition

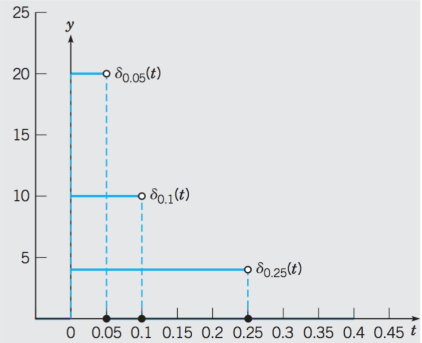
For $\epsilon > 0$, define the function

$$\delta_\epsilon(t) = \frac{u_0(t) - u_\epsilon(t)}{\epsilon} = \begin{cases} \frac{1}{\epsilon} & \text{if } 0 \le t < \epsilon \\ 0 & \text{if } t < 0 \text{ or } t \ge \epsilon \end{cases}$$

Graph of $\delta_\epsilon(t)$
for $\epsilon = 0.25$, 0.1 and 0.05.

For every $\epsilon > 0$ the area beneath the graph of $\delta_\epsilon(t)$ is 1.

The region beneath $\delta_\epsilon(t)$ is similar to the region beneath the graph of a force with large magnitude and active for short time $[0, \epsilon)$ when ϵ is very small.



- $\delta_\epsilon(t) = \dfrac{u_0(t) - u_\epsilon(t)}{\epsilon}$ approximates an impulse function acting in $[0, \epsilon)$ if ϵ is small.

- If ϵ is small and f is continuous on $[0, \epsilon)$, then $f(t) \approx f(0)$ for $t \in [0, \epsilon)$.
  Hence for every interval $[a, b]$ containing $[0, \epsilon)$:

$$\int_a^b f(t)\delta_\epsilon(t)\,dt = \frac{1}{\epsilon}\int_0^\epsilon f(t)\,dt \approx f(0)\frac{1}{\epsilon}\int_0^\epsilon dt = f(0)\,.$$

  The approximation is better the smaller ϵ is.

Wanted:

- The impulse should acting on $[t_0, t_0 + \epsilon)$ and not only on $[0, \epsilon)$ $\rightsquigarrow$ consider $\delta_\epsilon(t - t_0)$
- Take ϵ "as small as" possible $\rightsquigarrow$ take $\delta(t - t_0) \approx \lim_{\epsilon\to 0} \delta_\epsilon(t - t_0)$,

Here $\approx$ means that we look for a "function" (or a mathematical object) having the properties that $\lim_{\epsilon\to 0} \delta_\epsilon(t - t_0)$ would have (if the limit existed).

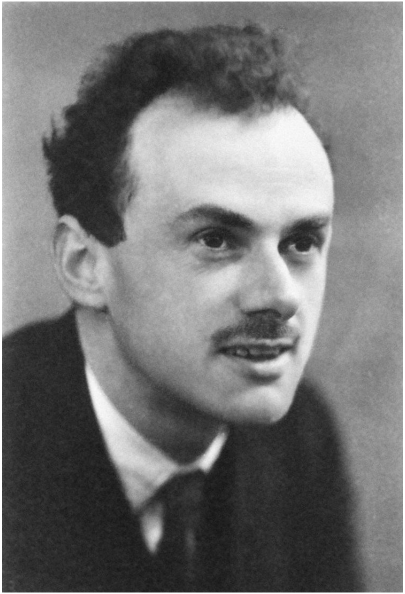
The "function" δ we are looking for is such that $\delta(t - t_0)$ should satisfy the following properties:

(a) $\delta(t - t_0) = \lim_{\epsilon \to 0} \delta_\epsilon(t - t_0) = 0$ for $t \neq t_0$.

(b) $\int_a^b f(t)\delta(t - t_0)dt = \lim_{\epsilon \to 0} \int_a^b f(t)\delta_\epsilon(t - t_0)dt = f(t_0)$
for any function f continuous on an interval $a \leq t_0 < b$.

There is no ordinary function satisfying these properties. Nevertheless, the above properties can define a mathematical object which goes under the name of a **generalized function** (or **distribution**).

δ defined by (1) and (2) is a generalized function. It is often known under the name of **delta function** (or **Dirac delta function** or **Dirac delta** or, more appropriately, **Dirac delta distribution**).



Paul Adrien Maurice Dirac (1902–1984).

English theoretical physicist.

Dirac made fundamental contributions to both quantum mechanics and quantum electrodynamics.

Nobel Prize in Physics in 1933

Since $t \mapsto e^{-st}$ is a continuous function of t for every fixed s, we obtain from the second property of δ:

If $t_0 \geq 0$ is arbitrarily fixed, then

$$\mathcal{L}\{\delta(t - t_0)\}(s) = \int_0^\infty e^{-st}\delta(t - t_0)dt = e^{-st_0} .$$

In particular, if $t_0 = 0$, then:

$$\mathcal{L}\{\delta(t)\}(s) = \int_0^\infty e^{-st}\delta(t)dt = 1 .$$

Example:
Find the solution of the following initial value problems:

- $y'' + 4y = \delta(t - 4\pi)$ with $y(0) = 0$ and $y'(0) = 0$
- $2y'' + y' + 6y = \delta(t - \pi/6) \sin t$ with $y(0) = 0$ and $y'(0) = 0$.