Section 5.8: Convolution integrals and their applications

Main Topics:

- **Convolution of functions**
- **Input-ouput problems**
- **Free response, forced response**
- **Transfer function, impulse response**
- **Applications to DE's**
- **Laplace transforms.**

Motivation: Convolution integrals as solutions of IVP's

Recall that the method of variation of parameters (Section 4.7).
It gives a a particular solution of $y'' + p(t)y' + q(t)y = g(t)$ as:

$$Y(t) = -y_1(t) \int \frac{y_2(t)g(t)}{W[y_1, y_2](t)}\, dt + y_2(t) \int \frac{y_1(t)g(t)}{W[y_1, y_2](t)}\, dt$$

where:

- y_1 and y_2 form a fundamental system of solutions of $y'' + p(t)y' + q(t)y = 0$.
- $W[y_1, y_2](t) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$.

$Y(t)$ as above is given up to constants (because of the indefinite integrals).
Fix the constant by replacing $\int$ by $\int_0^t$:

$$Y(t) = -y_1(t) \int_0^t \frac{y_2(\tau)g(\tau)}{W[y_1, y_2](\tau)}\, d\tau + y_2(t) \int_0^t \frac{y_1(\tau)g(\tau)}{W[y_1, y_2](\tau)}\, d\tau\, .$$

Example: $y'' + y = g(t)$ with $y(0) = y'(0) = 0$

The method of variation of parameters provides the solution in the integral form

$$y(t) = \int_0^t \sin(t - \tau)g(\tau)d\tau$$

Convolution Integrals

Definition (Definition 5.8.1)

Let $f(t)$ and $g(t)$ be two piecewise continuous functions on the interval $[0, +\infty)$. The **convolution of** f **and** g is defined by

$$h(t) = \int_0^t f(t-\tau)g(\tau)d\tau$$

This integral on the RHS is known as the **convolution integral**.
The convolution of f and g is also called the **convolution product** of f and g, denoted by $f \star g$.

The name "convolution product" is motivated by the following properties.

Theorem (Theorem 5.8.2)

(i) $f \star g = g \star f$ *(commutative law).*

(ii) $f \star (g_1 + g_2) = f \star g_1 + f \star g_2$ *(distributive law).*

(iii) $(f \star g) \star k = f \star (g \star k)$ *(associative law).*

(iv) $f \star 0 = 0 \star f = 0.$

Here 0 is the function equal to zero everywhere.

The convolution product is interesting when solving DE because of the following behaviour with respect to the Laplace transform.

Theorem (Theorem 5.8.3, Convolution Theorem)

Let $a \geq 0$. Suppose that both the Laplace transforms

$$F(s) = \mathcal{L}\{f\}(s) \qquad \text{and} \qquad G(s) = \mathcal{L}\{g\}(s)$$

exist for $s > a$.

Then the product function $H(s) = F(s)G(s)$ coincides with the Laplace transform $\mathcal{L}\{h\}(s)$ of the convolution product h of f and g. In other words,

$$\mathcal{L}\{f \star g\} = \mathcal{L}\{f\}\mathcal{L}\{g\}$$

Example:

(1) Find the Laplace transform of

$$f(t) = \int_0^t \sin(t-\tau)\cos(\tau)\, d\tau$$

(2) Find the inverse Laplace transform of

$$F(s) = \frac{1}{(s^2+1)^2}$$

The initial value problem

$$ay'' + by' + cy = g(t) \quad \text{with } y(0) = y_0 \text{ and } y'(0) = y_1$$

is called an **input-output problem**.

The function g is the **input** to the system.
The solution $y(t)$ of the IVP is called the **total response**.

Apply the Laplace transform and set $Y = \mathcal{L}\{y\}$ and $G = \mathcal{L}\{g\}$.
The IVP can restated as

$$(as^2 + bs + c)Y(s) - (as + b)y_0 - ay_1 = G(s)$$

i.e.

$$Y(s) = H(s)\Big((as + b)y_0 + ay_1\Big) + H(s)G(s)$$

where

$$H(s) = \frac{1}{as^2 + bs + c}$$

is called the **transfer function**.
Thus:

$$y(t) = \mathcal{L}^{-1}\{H(s)\big((as + b)y_0 + ay_1\big)\} + (h \star g)(t)$$

Look at $y(t) = \mathcal{L}^{-1}\{H(s)[(as+b)y_0 + ay_1]\} + (h \star g)(t)$.

The first term $\mathcal{L}^{-1}\{H(s)[(as+b)y_0 + ay_1]\}$ is the solution to the IVP

$$ay'' + by' + cy = 0 \quad \text{with } y(0) = y_0 \text{ and } y'(0) = y_1$$

The function $\mathcal{L}^{-1}\{H(s)[(as+b)y_0 + ay_1]\}$ is called the **free response** .

The second term $(h \star g)(t)$ is the solution to the IVP

$$ay'' + by' + cy = g(t) \quad \text{with } y(0) = y'(0) = 0$$

The function $(h \star g)(t)$ is called the **forced response** .

Thus: in the t-domain, the total response is the sum of the free response and the forced response.

Correpondingly, in the s-domain:
$Y(s) = H(s)[(as+b)y_0 + ay_1] + H(s)G(s)$ is the **total response**,
$H(s)[(as+b)y_0 + ay_1]$ is the **free response**,
$H(s)G(s)$ is the **forced response**.

An alternative description of the free response can be found by the methods of Chapter 4:

The free response $\mathcal{L}^{-1}\{H(s)[(as+b)y_0+ay_1]\}$ is the solution to the IVP

$$ay''+by'+cy=0 \quad \text{with } y(0)=y_0 \text{ and } y'(0)=y_1$$

It is therefore of the form $\alpha_1 y_1+\alpha_2 y_2$ where

- y_1, y_2 form a fundamental system of solutions of $ay''+by'+cy=0$
- α_1,α_2 are determined by the initial conditions.

The total response as the sum of the free response and the forced response.

	Total Response		Free Response		Forced Response
s-domain:	$Y(s)$	$=$	$H(s)[(as+b)y_0+ay_1]$	$+$	$H(s)G(s)$
t-domain:	$y(t)$	$=$	$\alpha_1 y_1(t)+\alpha_2 y_2(t)$	$+$	$\int_0^t h(t-\tau)g(\tau)\,d\tau$

$$\alpha_1=\frac{y_0y_2'(0)-y_1y_2(0)}{y_1(0)y_2'(0)-y_1'(0)y_2(0)} \qquad \text{and} \qquad \alpha_2=\frac{y_1y_1(0)-y_0y_1'(0)}{y_1(0)y_2'(0)-y_1'(0)y_2(0)}$$

Consider the transfer function $H(s) = \dfrac{1}{as^2 + bs + c}$.

It is the ratio of the forced response to the input in the s-domain.

Let $h(t) = \mathcal{L}^{-1}\{H(s)\}$.

Since $\mathcal{L}\{\delta(t)\} = 1$, the function $h(t)$ is the solution of the IVP

$$ay'' + by' + cy = \delta(t) \quad \text{with} \quad y(0) = y'(0) = 0$$

$h(t)$ is called the **impulse response**.

Example:
Find the transfer function, the impulse function, the forced response, the free response and the total response of the input-output problem

$$4y'' + 4y' + 17y = g(t) \quad \text{with} \quad y(0) = y'(0) = 0$$

where $g(t)$ is any piecewise continuous function of exponential order.